P–3 Mathematics

CA Preschool Learning Foundations

CA Common Core State Standards

Fourth Edition

SAN MATEO COUNTY OFFICE OF EDUCATION
Anne E. Campbell • County Superintendent of Schools

Includes excerpts from California Department of Education documents (1) California Preschool Learning Foundations, Volume 1(2008), (2) K-12 California's Common Core Content Standards for Mathematics (2010)

HEISING-SIMONS FOUNDATION

Preschool Learning Foundations and Common Core State Standards, Mathematics
Strand and Domain Alignment

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Table 1
Overview of the Alignment of the California Preschool Learning Foundations with Key Early Education Resources

Domains					
California Preschool Learning Foundations	**California Infant/Toddler Learning and Development Foundations**	**California Kindergarten Content Standards**	**Common Core State Standards**	**Head Start Child Development and Early Learning Framework**	**Additional Domains in the Head Start Child Development and Early Learning Framework with Corresponding Content**
Social–Emotional Development	Social–Emotional Development	Health Education Mental, Emotional, and Social Health		Social & Emotional Development	Approaches to Learning Logic & Reasoning
Language and Literacy	Language Development	English–Language Arts	English–Language Arts	Language Development Literacy Knowledge & Skills	
English—-Language Development	Language Development	English–Language Development		English Language Development	Literacy Knowledge & Skills
Mathematics	Cognitive Development	Mathematics	Mathematics	Mathematics Knowledge & Skills	Logic & Reasoning Approaches to Learning
Visual and Performing Arts	All Domains	Visual and Performing Arts		Creative Arts Expression	Logic & Reasoning
Physical Development	Perceptual and Motor Development Cognitive Development	Physical Education		Physical Development & Health	
Health	All Domains	Health Education		Physical Development & Health	
History–Social Science	Social–Emotional Development Cognitive Development	History–Social Science		Social Studies Knowledge & Skills	Social & Emotional Development
Science	Cognitive Development Language Development	Science		Science Knowledge & Skills	Approaches to Learning Logic & Reasoning

California Preschool Learning Foundations

Volume 1

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Introduction

The preschool learning foundations are a critical step in the California Department of Education's efforts to strengthen preschool education and school readiness and to close the achievement gap in California. They describe competencies—knowledge and skills—that most children can be expected to exhibit in a high-quality program as they complete their first or second year of preschool. In other words, the foundations describe what all young children typically learn with appropriate support.

The support young children need to attain the competencies varies from child to child. Many children learn simply by participating in high-quality preschool programs. Such programs offer children environments and experiences that encourage active, playful exploration and experimentation. With play as an integral part of the curriculum, high-quality programs include purposeful teaching to help children gain knowledge and skills. In addition, many children in California's preschools benefit from specific support in learning English. Other children may have a special need that requires particular accommodations and adaptations. To serve all children, preschool programs must work to provide appropriate conditions for learning and individually assist each child to move along a pathway of healthy learning and development.

All 50 states either have developed preschool standards documents or are in the process of doing so. Many of them have sought to align early learning standards with their kindergarten content standards. In most cases these alignment efforts have focused on academic content areas, such as English–language arts or mathematics. In California priority has been placed on aligning expectations for preschool learning with the state's kindergarten academic content standards and complementing the content areas with attention to social-emotional development and English-language development. Like the learning in such domains as language and literacy and mathematics, the concepts in social-emotional development and English-language development also contribute significantly to young children's readiness for school (*From Neurons to Neighborhoods* 2000; *Eager to Learn* 2000; *Early Learning Standards* 2002). Because the focus on preschool learning in California includes the full range of domains, the term "foundations" is used rather than "standards." This term is intended to convey that learning in every domain affects young children's readiness for school.

The preschool learning foundations presented in this document cover the following domains:

- Social-Emotional Development
- Language and Literacy
- English-Language Development (for English learners)
- Mathematics

Together, these domains represent crucial areas of learning and development for young children. The foundations within a particular domain provide a thorough overview of development in that domain. Preschool children can be considered from the perspective of one domain, such as language and literacy or social-emotional development. Yet, when taking an in-depth look at one domain, one needs to keep in mind that, for young children, learning is usually an integrated experience. For example, a young child may be concentrating on mathematical reasoning, but at the same time, there may be linguistic aspects of the experience.

The foundations written for each of these domains are based on research and evidence and are enhanced with expert practitioners' suggestions and examples. Their purpose is to promote understanding of preschool children's learning and to guide instructional practice. It is anticipated that teachers, administrators, parents, and policymakers will use the foundations as a springboard to augment efforts to enable all young children to acquire the competencies that will prepare them for success in school.

Overview of the Foundations

The strands for each of the domains discussed previously are listed in this section.

Social-Emotional Development Domain. The social-emotional development domain consists of the following three strands:

1. *Self,* which includes self-awareness and self-regulation, social and emotional understanding, empathy and caring, and initiative in learning
2. *Social Interaction,* which focuses on interactions with familiar adults, interactions with peers, group participation, and cooperation and responsibility
3. *Relationships,* which addresses attachments to parents, close relationships with teachers and caregivers, and friendships

The competencies covered by the social-emotional development foundations underscore the multiple ways in which young children's development in this domain influences their ability to adapt successfully to preschool and, later on, in school.

Language and Literacy Domain. The language and literacy foundations address a wide range of specific competencies that preschool children will need support to learn. These foundations focus on the following three strands:

1. *Listening and Speaking,* which includes language use and conventions, vocabulary, and grammar
2. *Reading,* which covers concepts about print, phonological awareness, alphabetics and word/print

recognition, comprehension and analysis of age-appropriate text, and literacy interest and response

3. *Writing,* which focuses on writing strategies, including the emergent use of writing and writing-like behaviors

The foundations that were written for this domain reflect the field's growing interest in and understanding of the knowledge and skills that foster children's language and literacy learning during the preschool years.

English-Language Development Domain. The English-language development foundations are specifically designed for children entering preschool with a home language other than English. Some English learners will begin preschool already having had some experience with English. For other English learners, preschool will offer them their first meaningful exposure to English. No matter how much background English learners have with English before they enter preschool, they will be on a path of acquiring a second language. As the English-language development foundations indicate, the learning task for English learners is sequential and multifaceted. English learners will need support in developing knowledge and skills in the following four strands:

1. *Listening,* which includes understanding words, requests and directions, and basic and advanced concepts
2. *Speaking,* which focuses on using English to communicate needs, expand vocabulary, become skillful at engaging in conversations, use increasingly complex grammatical constructions when speaking, understand grammar, ask questions, use social conventions, and tell personal stories
3. *Reading,* which covers appreciating and enjoying reading, understanding book reading, understanding print conventions, demonstrating awareness that print conveys meaning, developing awareness and recognition of letters, demonstrating phonological awareness, and manipulating sounds, such as rhyming
4. *Writing,* which includes understanding the communicative function of writing and engaging in simple writing and writing-like behaviors

Unlike the three other sets of foundations, in which the foundations are linked to age, the English-language development foundations are defined by three levels of development—Beginning, Middle, and Later. Depending on their prior experience with using their home language and English to communicate with others, preschool English learners will go through these levels at different paces. Once children reach the Later level, they will still need support to continue acquiring English and to apply their developing linguistic abilities in every domain.

Mathematics Domain. Young children's development of mathematics knowledge and skills is receiving increasing attention in research and practice. The mathematics foundations cover the following five strands:

1. *Number Sense,* which includes understanding of counting, number relationships, and operations
2. *Algebra and Functions (Classification and Patterning),* which focuses on sorting and classifying objects

and recognizing and understanding simple, repeating patterns

3. *Measurement*, which includes comparison and ordering
4. *Geometry*, which focuses on properties of objects (shape, size, position) and the relation of objects in space
5. *Mathematical Reasoning*, which addresses how young children use mathematical thinking to solve everyday problems

Preschool programs can promote young children's learning in this domain by encouraging children to explore and manipulate materials that engage them in mathematical thinking and by introducing teacher-guided learning activities that focus on mathematical concepts.

Organization of the Foundations

In the main body of this document, each strand is broken out into one or more substrands, and the foundations are organized under the substrands. Foundations are presented for children at around 48 months of age and at around 60 months of age. In some cases the difference between the foundations for 48 months and 60 months is more pronounced than for the other foundations. Even so, the foundations focus on 48 and 60 months of age because they correspond to the end of the first and second years of preschool. Of course, teachers need to know where each child is on a continuum of learning throughout the child's time in preschool. The Desired Results Developmental Profile-Revised (DRDP-R) is a teacher observation tool that is being aligned with the foundations. The DRDP-R gives teachers a means to observe children's learning along a continuum of four developmental levels.

Finally, the examples listed under each foundation give a range of possible ways in which children can demonstrate a foundation. The examples suggest different kinds of contexts in which children may show the competencies reflected in the foundations. Examples highlight that children are learning while they are engaging in imaginative play, exploring the environment and materials, making discoveries, being inventive, or interacting with teachers or other adults. Although often illustrative of the diversity of young children's learning experiences, the examples listed under a foundation are not exhaustive. In fact, teachers often observe other ways in which young children demonstrate a foundation.

Note: The Appendix, "The Foundations," contains a listing of the foundations in each domain, without examples.

Universal Design for Learning

The California preschool learning foundations are guides to support preschool programs in their efforts to foster the learning and development of all young children in California, including children who have disabilities. In some cases, children with disabilities will need to use alternate methods for demonstrating their development. It is important to provide opportunities to follow different pathways to learning in the preschool foundations in order to make them helpful for all of California's children. To that end, the California preschool learning foundations incorporate a concept known as universal design for learning.

Developed by the Center for Applied Special Technology (CAST), universal design for learning is based on the realization that children learn in different ways (CAST 2007). In today's diverse preschool settings and programs, the use of a curriculum accessible to all learners is critical to successful early learning. Universal design for learning is not a single approach that will accommodate everyone; rather, it refers to providing multiple approaches to learning in order to meet the needs of diverse learners. Universal design provides for multiple means of representation, multiple means of engagement, and multiple means of expression (CAST 2007). Multiple means of representation refers to providing information in a variety of ways so the learning needs of all of the children are met. Multiple means of expression refers to allowing children to use alternative methods to demonstrate what they know or what they are feeling. Multiple means of engagement refers to providing choices for activities within the setting or program that facilitate learning by building on children's interests.

The examples given in the preschool learning foundations have been worded in such a way as to incorporate multiple means of receiving and expressing. This has been accomplished by the inclusion of a variety of examples for each foundation and the use of words that are inclusive rather than exclusive, as follows:

- The terms "communicates" and "responds" are often used rather than the term "says." "Communicates" and "responds" are inclusive of any language and any form of communication, including speaking, sign language, finger spelling, pictures, electronic communication devices, eye-pointing, gesturing, and so forth.
- The terms "identifies" and "indicates or points to" are often used to represent multiple means of indicating objects, people, or events in the environment. Examples include, among other means of indicating, the use of gestures, eye-pointing, nodding, or responding "yes" or "no" when another points to or touches an object.

Teachers should read each foundation and the accompanying examples, then consider the means by which a child with a disability might best acquire information and demonstrate competence in these areas. A child's special education teacher, parents, or related service provider may be contacted for consultation and suggestions.

The Foundations and Preschool Learning in California

The foundations are at the heart of the CDE's approach to promoting preschool learning. Teachers use best practices, curricular strategies, and instructional techniques that assist children in learning the knowledge and skills described in the preschool learning foundations. The "how to's" of teaching young children include setting up environments, supporting children's self-initiated play, selecting appropriate materials, and planning and implementing teacher-guided learning activities. Two major considerations underlie the "how to's" of teaching. First, teachers can effectively foster early learning by thoughtfully considering the preschool learning foundations as they plan environments and activities. And second, during every step in the planning for young children's learning, teachers have an opportunity to tap into the prominent role of play. Teachers can best support young children both by encouraging the rich learning that occurs in children's self-initiated play and by introducing purposeful instructional activities that playfully engage preschoolers in learning.

Professional development is a key component in fostering preschool learning. The foundations can become a unifying element for both preservice and in-service professional development. Preschool program directors and teachers can use the foundations to facilitate curriculum planning and implementation. At the center of the CDE's evolving system for supporting young children during the preschool years, the foundations are designed to help teachers be intentional and focus their efforts on the knowledge and skills that all young children need to acquire for success in preschool and, later on, in school.

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FOUNDATIONS IN

Mathematics

The preschool learning foundations identify for teachers and other educational stakeholders a set of behaviors in mathematics learning that are typical of children who will be ready to learn what is expected of them in kindergarten. The foundations provide age-appropriate competencies expected for *older* three-year-olds (i.e., at around 48 months of age) and for *older* four-year-olds (i.e., at around 60 months of age). That is, the preschool learning foundations represent goals to be reached by the time a three-year-old is just turning four and a four-year-old is just turning five. Focusing on the child's readiness for school in the domain of mathematics learning acknowledges that there must also be appropriate social-emotional, cognitive, and language development as well as appropriate motivation. Many such complementary and mutually supporting aspects of the child's overall development are addressed in the preschool learning foundations for other domains (e.g., social-emotional development, language and literacy, and English-language development).

These preschool learning foundations are designed with the assumption that children's learning takes place in everyday environments: through interactions, relationships, activities, and play that are part of a beneficial preschool experience. The foundations are meant to describe what is typically expected to be observed from young children in their everyday contexts, under conditions appropriate for healthy development, not as aspirational expectations under the best possible conditions. They are meant as guidelines and tools to support teaching, not as a list of items to be taught as isolated skills and not to be used for assessment.

Some mathematics foundations mention specific expectations, using exact numbers to describe a counting range (e.g., "up to three," or "up to four") at different ages or to set a minimum criterion for a particular area (e.g., "compare two groups of up to five objects"). However, some children may exhibit competencies that go beyond the level described in a particular foundation, while others may need more time to reach that level. The foundations are meant to give teachers a general idea of what is typically expected from children at around 48 or 60 months of age and are not intended to set limits on the way teachers support children's learning at different levels.

Children with special needs can demonstrate mathematical knowledge in various ways and do not necessarily need to engage in motor behavior. For example, a child might indicate to an adult or another child where to place each object in a sorting task. Or a child might ask a teacher to place objects in a particular order to make a repeating pattern. Children with visual impairments might be offered materials for counting, sorting, problem solving, and so forth that are easily distinguishable by touch. Any means of expression and engagement available to the child should be encouraged.

Organization of the Mathematics Foundations

The California preschool learning foundations in mathematics cover five main developmental strands: Number Sense, Algebra and Functions (Classification and Patterning), Measurement, Geometry, and Mathematical Reasoning. These strands were identified after a careful review of research, the *Principles and Standards for School Mathematics* (NCTM 2000), and the California mathematics content standards for kindergarten through grade twelve (K–12).

The preschool mathematics foundations expand on the standards identified by the NCTM (2000) for the preschool age to include more detailed, age-specific expectations in the key mathematics content areas. In addition, the preschool foundations for mathematics are closely aligned with the California K–12 mathematics content standards, yet there are some particular differences in the organization of the mathematics strands. In the preschool learning foundations, Measurement and Geometry are two separate strands rather than one combined strand of Measurement and Geometry. Also, the preschool learning foundations, unlike the K–12 mathematics content standards, do not include a separate strand for statistics, data analysis, and probability. The foundations for Patterning are included in the strand for Algebra and Functions, along with the foundations for Classification.

The numbering system for the mathematics foundations follows the same numbering system used in the California K–12 mathematics content standards. The major divisions within a strand are referred to as substrands and are numbered 1.0, 2.0, and so forth. Each substrand is divided into a column for children "around 48 months of age" and a column for children "around 60 months of age" on each page. The description for younger preschool children is different from the one for older preschool children. The separate foundations are written under their substrand column by age range and are numbered sequentially. Where a substrand is numbered 1.0, the foundations under the substrand would be 1.1, 1.2, and so forth, where a substrand is numbered 2.0, the foundations under the substrand would be 2.1, 2.2, and so forth for both columns.

Immediately below each foundation, a few examples are given. The examples are meant to clarify the foundation by illustrating how the competency described in the foundation might be observed in the preschool environment. They are *not* meant to be used as a checklist of the knowledge and skills that a child must demonstrate before the teacher can decide that a competency is present.

A developmental progression by age range is articulated within each substrand. That is, the substrand description and foundations for children at around 60 months of age are written to indicate a higher level of development than the foundations for children at around 48 months of age in that same substrand. For some foundations, the change between 48- and 60-month-old children is more pronounced than for other foundations. Although there is a developmental progression from around 48 months of age to around 60 months of age within a substrand, the order in which the strands are presented is not meant to indicate any sense of developmental progression from strand to strand or from substrand to substrand within a strand.

At the end of the foundations, bibliographic notes provide a review of the research base for the foundations. Following the bibliographic notes, a list of references for the entire set of mathematics foundations is provided. Brief explanations of each strand are as follows:

Number Sense—important aspects of counting, number relationships, and operations

Preschool children develop an initial qualitative understanding of a quantity of small groups of objects without actually counting the objects. This understanding is referred to as visually knowing or "subitizing." It supports the ability to compare small groups of objects: to know if the groups are the same, if one group is larger (smaller), or which has more (fewer). Also developing is the ability to approach simple arithmetic-like operations on groups of objects with ideas such as "adding to," "putting together," "taking apart," "taking away," and so forth. Preschool is the time when children learn to recite the numbers in order, recognize numerals, and begin to incorporate the idea of one-to-one-correspondence and true counting. This is also a time when preschool children begin to learn about cardinality, which is the concept of knowing the last number named is the quantity of objects counted.

Algebra and Functions (Classification and Patterning)—sorting and classifying objects; recognizing, extending, and creating patterns

Classification involves sorting, grouping, or categorizing objects according to established criteria. Analyzing, comparing, and classifying objects provide a foundation for algebraic thinking. Although preschool children may not know how all the objects in a mixed set can be sorted or be able to say much about why some objects go together, they do begin to group like with like at around 48 months of age and will do so more completely at around 60 months of age. These foundations use the idea of sorting objects by some attribute. The term "attribute" is used here to indicate a property of objects, such as color or shape, that would be apparent to a preschooler and that the preschooler could use as a basis for grouping or sorting. A younger preschool child is expected to show some sorting of a group of objects, but not necessarily do so completely or without errors. A young preschooler might sort farm animals but remove only the cows and leave the rest ungrouped, and there may be a pig or two or a horse mixed in. But an older preschool child might make a group of all cows and a group of all pigs and a group of all horses. This competency is the precursor to many important mathematics abilities that will come later (e.g., the logic of what belongs in a set and what does not, grouping terms in an algebraic expression, data analysis, and graphing). Sorting and grouping in preschool will help prepare children for those later steps. Researchers Seo and Ginsburg (2004) point out that preschool children do not often spontaneously choose to do sorting activities on their own. Therefore, sorting is an area in which teacher facilitation and modeling across a range of situations and contexts is particularly important. The teacher should note that how a child sorts depends on the situation and the child's perception of the activity.

Thinking about patterns is another important precursor for learning mathematics in general and for learning algebra in particular (Clements 2004a). During the preschool years, children develop their abilities to recognize, identify, and duplicate patterns and to extend and create simple repeating patterns. Although less research has been conducted for preschoolers in patterning than in other areas, such as numbers and counting, recent studies (Klein and Starkey 2004; Starkey, Klein, and Wakelely 2004) provide information about the development of patterning skills. Children first learn to identify the core unit in a repeating pattern. Once they are able to identify the initial unit of a pattern, they can extend a pattern by predicting what comes next. Teacher facilitation and modeling are particularly important in introducing the notion of patterns, extending it to more aspects of the child's environment and daily activities, and encouraging the child's attempts to create patterns.

Measurement—comparing and ordering objects by length, weight, or capacity; precursors of measurement

Measuring is assigning a number of units to some property, such as length, area, or weight, of an object. Although much more learning will take place later as children become increasingly competent with core measurement concepts, preschool is when children gain many of the precursors to this kind of understanding about comparing, ordering, and measuring things. For example, young preschool children are becoming aware that objects can be compared by weight, height, or length and use such words as "heavier," "taller," or "longer" to make comparisons. They begin to compare objects directly to find out which is heavier, taller, and so forth. They can compare length by placing objects side by side and order three or more objects by size. By the time children are around 60 months old, they develop the understanding that measuring length involves repeating equal-size units and counting the number of units. They may start measuring length by laying multiple copies of same-size units end to end (Clements 2004a).

Geometry—properties of objects (shape, size, position) and the relation of objects in space

Geometry is a tool for understanding relations among shapes and spatial properties mathematically. Preschool children learn to recognize and name two-dimensional shapes, such as a circle, square, rectangle, triangle, and other shapes. At first, they recognize geometric shapes by their overall holistic physical appearance. As they gain more experience comparing, sorting, and analyzing shapes, children learn to attend to individual attributes and characteristics of different shapes. Younger preschool children use shapes in isolation, while older preschool children use shapes to create images of things they know and may combine shapes into new shapes (Clements 2004a, 2004b). In the early preschool years, children also develop spatial reasoning. They can visualize shapes in different positions and learn to describe the direction, distance, and location of objects in space. Teachers can facilitate children's development of geometry and spatial thinking by offering many opportunities to explore attributes of different shapes and to use vocabulary words about the position of objects in space.

Mathematical Reasoning—using mathematical thinking to solve problems in play and everyday activities

Children in preschool encounter situations in play and everyday activities that require them to adapt and change their course of action. Although they may not realize it, some situations call for mathematical reasoning—to determine a quantity (e.g., how many spoons?) or to reason geometrically (e.g., what shape will fit?). Other situations require general reasoning. For preschoolers, when the context is familiar and comfortable enough, a simple strategy may be applied to solve an immediate problem—even something as simple as counting the number of objects held in the hand or carrying a block over to see if there are others like it. A young preschool child may begin this process by trying a strategy that is not always effective. An older preschool child may try several strategies, finally finding one that works. The important point is that both younger and older preschool children learn through reasoning mathematically. As the above examples suggest, encouraging young children to engage in mathematical reasoning is not only beneficial in itself, it also opens the door to children's exploration of the other mathematics foundations, such as geometric shapes, counting, and classification.

MATHEMATICS

Number Sense*

At around 48 months of age	*At around 60 months of age*
1.0 Children begin to understand numbers and quantities in their everyday environment.	**1.0 Children expand their understanding of numbers and quantities in their everyday environment.**
1.1 Recite numbers in order to ten with increasing accuracy.†	**1.1** Recite numbers in order to twenty with increasing accuracy.†
Examples	**Examples**
• Recites one to ten incompletely or with errors while playing (e.g., "one, two, three, four, five, seven, ten"). • Recites one to ten while walking. • Recites one to ten while singing.	• Recites one to twenty incompletely or with errors (e.g., "one, two, three, four, five, . . . nine, ten, eleven, twelve, thirteen, fifteen , seventeen, eighteen, twenty"). • Chants one to twenty in order while swinging. • Recites one to twenty to show her friend how high she can count.
1.2 Begin to recognize and name a few written numerals.	**1.2** Recognize and know the name of some written numerals.
Examples	**Examples**
• Communicates, "That's a one," when playing with magnetic numerals. • Indicates or points to the numerals on a cube and names, "three, two, five." • Identifies the numeral 3 on the page of the *Five Little Speckled Frogs* book while sitting with a teacher.	• Names some numerals found in books or during a game. • Points to numerals in a number puzzle as the teacher names them.

* Throughout these mathematics foundations many examples describe the child manipulating objects. Children with motor impairments may need assistance from an adult or peer to manipulate objects in order to do things such as count, sort, compare, order, measure, create patterns, or solve problems. A child might also use adaptive materials (e.g., large manipulatives that are easy to grasp). Alternately, a child might demonstrate knowledge in these areas without directly manipulating objects. For example, a child might direct a peer or teacher to place several objects in order from smallest to largest. Children with visual impairments might be offered materials for counting, sorting, or problem solving that are easily distinguishable by touch. Their engagement is also facilitated by using containers, trays, and so forth that contain their materials and clearly define their work space.

† Some children may not be able to count by either saying the numbers or signing them. Any means available to the child for demonstrating knowledge of numbers in order should be encouraged. For example, a child may indicate or touch number cards or might respond yes or no when an adult counts.

At around 48 months of age	*At around 60 months of age*
1.3 Identify, without counting, the number of objects in a collection of up to three objects (i.e., subitize).	**1.3** Identify, without counting, the number of objects in a collection of up to four objects (i.e., subitize).
Examples	**Examples**
• Perceives directly (visually, tactilely, or auditorily) the number of objects in a small group without needing to count them. • Indicates or points to a pile of blocks and communicates, “Three of them.” • Attends to the child next to her at snack time and communicates, “Clovey has two.” • Looks briefly at a picture with three cats and immediately communicates the quantity by saying “three” or showing three fingers.	• Perceives directly (visually, tactilely, or auditorily) the number of objects in a small group without needing to count them. • Looks briefly at a picture of four frogs and immediately communicates the quantity four. • During storytime, puts her hand on the picture of four ladybugs and communicates, “Four ladybugs.” • Correctly points out, “That’s three cars there.”
1.4 Count up to five objects, using one-to-one correspondence (one object for each number word) with increasing accuracy.*	**1.4** Count up to ten objects, using one-to-one correspondence (one object for each number word) with increasing accuracy.*
Examples	**Examples**
• After building a block tower, counts the number of blocks by pointing to the first block and communicating “one,” then pointing to the next block and communicating “two.” The child counts up to five blocks. • Indicates or points to each toy in a line while communicating, “One, two, three, four, five.”	• Indicates or points to a flower in the garden and communicates, “one,” then points to another flower and communicates, “two.” The child counts up to seven different flowers. • Counts ten children by identifying them one by one during circle time. • Counts the blocks in a pile, keeping track of which blocks have already been counted. • Counts out eight napkins in preparation for snack time.

* Children with motor disabilities may need assistance manipulating objects in order to count them. Children may also demonstrate knowledge of object counting by using eye-pointing or by counting while an adult or another child touches or moves the objects.

At around 48 months of age	At around 60 months of age
1.5 Use the number name of the last object counted to answer the question, "How many . . . ?"	**1.5** Understand, when counting, that the number name of the last object counted represents the total number of objects in the group (i.e., cardinality).
Examples	**Examples**
• Counts the number of sticks in her hand, communicating, "one, two, three, four, five." The teacher asks, "How many sticks do you have?" and the child communicates "five." • When asked, "How many cars do you have?" counts, "one, two three, four" and communicates, "four." • Counts the beads in her necklace, communicating, "one, two, three, four, five, six." A friend asks, "How many beads do you have?" and the child replies, "six."	• After giving away some bears, counts the remaining bears to find out how many are left and communicates, "I now have six bears." • Lines up cars on a track and counts, then communicates, "My train has seven cars!" • Counts dolls, "one, two, three, four" and communicates, "There are four dolls." • Counts her sticks and communicates, "I have five," when the teacher asks during an activity, "Does everyone have five sticks?" • Counts five apple slices and recognizes there is one slice of apple for each of the five children around the table.

At around 48 months of age	*At around 60 months of age*
2.0 Children begin to understand number relationships and operations in their everyday environment.	**2.0 Children expand their understanding of number relationships and operations in their everyday environment.**
2.1 Compare visually (with or without counting) two groups of objects that are obviously equal or nonequal and communicate, "more" or "same."*	**2.1** Compare, by counting or matching, two groups of up to five objects and communicate, "more," "same as," or "fewer" (or "less").*
Examples	**Examples**
• Examines two groups of counting bears, one with two bears and the other with six bears, and indicates or points to the group of six bears when asked which group has more. • Communicates, "I want more—she's got more stamps than me" during a small group activity. • Communicates, "We have the same," when referring to apple slices during snack time.	• Counts the number of rocks he has and the number a friend has and communicates, "Five and five, you have the same as me." • Compares a group of four bears to a group of five bears and communicates, "This one has less." • Counts her own sand toys, then counts a friend's and communicates, "You have more."
2.2 Understand that adding to (or taking away) one or more objects from a group will increase (or decrease) the number of objects in the group.	**2.2** Understand that adding one or taking away one changes the number in a small group of objects by exactly one.
Examples	**Examples**
• Has three beads, takes another, and communicates, "Now I have more beads." • When the teacher adds more cats on the flannel board, indicates that there are now more cats. • While playing bakery, communicates that after selling some bagels there are now fewer bagels in the bakery shop. • Gives away two dolls and communicates that now she has fewer.	• Adds another car to a pile of five to have six, just like his friend. • Removes one animal from a collection of eight animals and communicates, "She has seven now." • Correctly predicts that if one more car is added to a group of four cars, there will be five.

*Comparison may be done visually, tactilely, or auditorily.

At around 48 months of age	At around 60 months of age
2.3 Understand that putting two groups of objects together will make a bigger group.	**2.3** Understand that putting two groups of objects together will make a bigger group and that a group of objects can be taken apart into smaller groups.
Examples	**Examples**
• Combines his blocks with a pile of his friend's blocks and communicates, "Now we have more." • Puts together crayons from two separate boxes to have more. • Puts together the red bears and the yellow bears to have a bigger group of bears.	• Refers to a collection of six balloons and communicates, "Three red balloons for me and three green ones for you." • Indicates seven by holding up five fingers on one hand and two fingers on another. • Removes three (of five) ducks from the flannel board, communicating, "Three left, and only two stay" when acting a story.
2.4 Solve simple addition and subtraction problems nonverbally (and often verbally) with a very small number of objects (sums up to 4 or 5).	**2.4** Solve simple addition and subtraction problems with a small number of objects (sums up to 10), usually by counting.
Examples	**Examples**
• Recognizes that one ball together with another one makes a total of two balls. The child may create a matching collection or say or indicate "two." • Adds one car to a train with two cars and indicates the total number of cars in train by showing three fingers. • Recognizes that only two bananas are left after giving away one of three bananas to a friend. • Takes away one flower from a group of four flowers on the flannel board, while acting out a story, and communicates that only three flowers are left.	• During a small group activity, count oranges on the flannel board and communicate, "There are six oranges." The teacher puts one more orange on the board and asks, "How many oranges do we have now?" Some say seven; others first count, "One, two three, four, five, six, seven" and then say seven. • Adds two more cups to a group of two, says that there are four cups. • Takes two boats away from a group of five boats and communicates, "One, two, three—three boats left" while playing with friends. • Watches a friend connect a train with three cars to a second train with three cars. Counts the cars and communicates, "Now our train has six cars." • Builds a stack of five blocks and adds two more saying, "One, two, three, four, five, six, seven. I have seven blocks now."

Algebra and Functions
(Classification and Patterning)*

At around 48 months of age	*At around 60 months of age*
1.0 Children begin to sort and classify objects in their everyday environment.	**1.0 Children expand their understanding of sorting and classifying objects in their everyday environment.**
1.1 Sort and classify objects by *one* attribute into two or more groups, with increasing accuracy.	**1.1** Sort and classify objects by *one or more* attributes, into two or more groups, with increasing accuracy (e.g., may sort first by one attribute and then by another attribute).†
Examples	**Examples**
• Selects some red cars for himself and some green cars for his friend, leaving the rest of the cars unsorted. • Chooses the blue plates from a variety of plates to set the table in the kitchen play area. • Sorts through laundry in the basket and takes out all the socks. • Places all the square tiles in one bucket and all the round tiles in another bucket. • Attempts to arrange blocks by size and communicates, "I put all the big blocks here and all the small ones there."	• Sorts the large blue beads into one container and the small red beads in another. • Puts black beans, red kidney beans, and pinto beans into separate bowls during a cooking activity. • Arranges blocks on the shelf according to shape. • Sorts a variety of animal photographs into two groups: those that fly and those that swim. • Sorts buttons first by size and then each subgroup by color into muffin tin cups.

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† Attributes include, but are not limited to, size, shape, or color.

MATHEMATICS

At around 48 months of age	*At around 60 months of age*
2.0 Children begin to recognize simple, repeating patterns.*	**2.0 Children expand their understanding of simple, repeating patterns.***
2.1 Begin to identify or recognize a simple repeating pattern.	**2.1** Recognize and duplicate simple repeating patterns.
Examples	**Examples**
• Recognizes a simple repeating pattern made with interlocking cubes, such as yellow, green, yellow, green. • Sings, moves, or claps through part of a pattern song (e.g., the teacher begins a "clap-pat-clap-pat" pattern, and the child repeats with guidance). • Anticipates a repeating pattern in a storybook, with support.	• Fills in an item missing from a pattern (e.g., apple, pear, apple, pear), with guidance. • Copies simple repeating patterns, using the same kind of objects as the original pattern. • Attempts to sing, sign, move, or clap through a pattern song, trying to maintain the pattern.
2.2 Attempt to create a simple repeating pattern or participate in making one.	**2.2** Begin to extend and create simple repeating patterns.
Examples	**Examples**
• Puts together connecting blocks in alternating colors to form a repeating pattern, with guidance. • Demonstrates a pattern of claps, signs, or movements, with guidance. • Lines up pretzel sticks and cheese slices to make patterns at snack time.	• Adds a red bead and then a blue bead in a red-blue-red-blue pattern to complete a bead necklace. • Alternates short and tall blocks to make a fence around a farm. • Makes up a clapping or action pattern, "clap, clap, hop, hop" in rhythm to a song. • Uses different materials such as buttons, beads, or sequins to create patterns.

* A simple repeating pattern has two repeating elements. Examples are as follows: A-B-A-B (e.g., red-blue-red-blue); A-A-B-B (e.g., dog-dog-cat-cat); A-B-B-A-B-B (e.g., clap-stomp-stomp-clap-stomp-stomp); and so forth.

Measurement*

At around 48 months of age	*At around 60 months of age*
1.0 Children begin to compare and order objects.	**1.0 Children expand their understanding of comparing, ordering, and measuring objects.**
1.1 Demonstrate awareness that objects can be compared by length, weight, or capacity, by noting gross differences, using words such as *bigger, longer, heavier,* or *taller,* or by placing objects side by side to compare length.	**1.1** Compare two objects by length, weight, or capacity directly (e.g., putting objects side by side) or indirectly (e.g., using a third object).
Examples	**Examples**
• Communicates, "I'm big like my daddy." • Communicates, "This one's heavier" when choosing from a variety of beanbags in a basket. • Communicates, "He has more clay than me." • Communicates, "Mine is longer than yours" when placing trains side by side to check which is longer. • Builds a tower beside another child, attempting to make her tower taller.	• Tries to determine if he is taller than another child by standing next to the child. • Uses a balance scale to find out which of two rocks is heavier. • Pours water into different size containers at the water table to find out which one holds more. • Shows that the blue pencil is longer than the red pencil by placing them side by side. • Compares the length of two tables by using a string to represent the length of one table and then laying the string against the second table. • Uses a paper strip to mark the distance from knee to foot and compares it to the distance from elbow to fingertip.

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At around 48 months of age	*At around 60 months of age*
1.2 Order three objects by size.	**1.2** Order four or more objects by size.
Examples	**Examples**
• Sets bowls by size in dramatic play area, the biggest bowl for daddy bear, the medium bowl for mommy bear, and the smallest bowl for baby bear. • Lines up three animal figures by size. • Attempts to arrange nesting cups or ring stackers in correct order by size.	• Arranges four dolls from smallest to largest in pretend play with dolls. • In sandbox, lines up buckets by size, from the bucket that holds the most sand to one that holds the least. • On a playground, orders different kinds of balls (e.g., beach ball, basketball, soccer ball, tennis ball) by size.
	1.3 Measure length using multiple duplicates of the same-size concrete units laid end to end.*
	Examples
	• Uses paper clips laid end to end to measure the length of different size blocks, with adult guidance. • Measures the length of a rug by laying same-size block units end to end and communicating, “The rug is ten blocks long,” with adult guidance. • Measures the length of a table using inch “worms,” with adult guidance. • Measures the distance from the reading area to the block area by using meter sticks, with adult guidance.

* A foundation for measurement is written only for children at around 60 months of age, because the development of the ability to use same-size units to measure quantity typically occurs between 48 months and 60 months of age.

Geometry*

At around 48 months of age	*At around 60 months of age*
1.0 Children begin to identify and use common shapes in their everyday environment.	**1.0 Children identify and use a variety of shapes in their everyday environment.**
1.1 Identify simple two-dimensional shapes, such as a circle and square.	**1.1** Identify, describe, and construct a variety of different shapes, including variations of a circle, triangle, rectangle, square, and other shapes.
Examples	**Examples**
• When playing a matching game, communicates, "This is a circle." • While playing shape bingo, indicates or points to the correct shape. • Indicates a shape block and communicates, "This is a square." • Sorts shape manipulatives of varying sizes into different shape groups (e.g., points to the group of triangles and communicates, "Here are the triangles: big, small, and very small triangles").	• While playing the "I Spy the Shape" game, communicates, "I see a circle—the clock." Later, says, "I see a rectangle—the table." • Correctly identifies shapes as the teacher calls them out in a game of shape bingo. • Uses play dough to construct rectangles of different sizes and orientations. • Sorts manipulatives of different sizes and orientations by shape and explains why a particular shape does or does not belong in a group. • Tears paper shape and communicates, "Look! A triangle" while making a collage.
1.2 Use individual shapes to represent different elements of a picture or design.	**1.2** Combine different shapes to create a picture or design.
Examples	**Examples**
• Uses a circle for a sun and a square for a house in a picture. • Puts together a foam shape puzzle in which each shape is outlined. • Creates a design by putting shape tiles together.	• Uses a variety of shapes to construct different parts of a building. • Uses flannel pieces of different shapes to create a design. • Creates a house, from different shapes, using a computer program.

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At around 48 months of age	*At around 60 months of age*
2.0 Children begin to understand positions in space.	**2.0 Children expand their understanding of positions in space.**
2.1 Identify positions of objects and people in space, such as in/on/under, up/down, and inside/outside.	**2.1** Identify positions of objects and people in space, including in/on/under, up/down, inside/outside, beside/between, and in front/behind.
Examples	**Examples**
• Goes under the table when the teacher communicates, " Pick up the cup. It's under the table." • Communicates to another child in the playhouse, "Put the pan on the stove." • Requests that another child put the balls inside the box. • Looks up when the teacher says, "If you look up, you'll see your coat."	• During a treasure hunt, gives or follows directions to find something behind the doll bed or under the mat. • Follows directions when asked by the teacher to stand in front of or behind another child. • Communicates, "Where's my book?" A friend says, "It's over there on the table." She finds the book. • Follows along with the directions during a game of "Simon Says" (e.g., "Put your hands in front of your legs").

Mathematical Reasoning*

At around 48 months of age	*At around 60 months of age*
1.0 Children use mathematical thinking to solve problems that arise in their everyday environment.	**1.0 Children expand the use of mathematical thinking to solve problems that arise in their everyday environment.**
1.1 Begin to apply simple mathematical strategies to solve problems in their environment.	**1.1** Identify and apply a variety of mathematical strategies to solve problems in their environment.
Examples	**Examples**
• Reconfigures blocks to build a balanced, tall tower by placing the rectangular blocks at the bottom and triangular blocks at the top. • Asks for one more paintbrush so he can put one brush in each paint cup while helping to set up an easel for painting. • Gives a friend two flowers and keeps two for himself, so they both have the same number of flowers. • Compares the length of her shoe to her friend's shoe by placing them side by side to check who has a longer shoe. • Classifies objects according to whether they can roll or not. • Pours sand from a big bucket to a smaller bucket and realizes that not all the sand can fit. The child looks for a bigger bucket.	• After placing plates and napkins around the snack table, recognizes that he needs one more napkin for the last place and asks the teacher for another napkin. • Following a discussion about the size of the room, works with other children to measure the length of the room using block units, lay blocks of the same size along the wall end to end, and count the number of blocks. • Predicts the number of small balls in a closed box and then communicates, "Let's count." • Has run out of long blocks to complete a road and solves the problem by using two smaller blocks to "fill in" for a longer block. • When in need of six cones to set up an obstacle course but having only four, communicates, "I need two more cones." • Sorts the animal figures into two groups, wild animals for him and pets for his friend, when asked to share the animal figures with a friend.

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Bibliographic Notes

Number Sense

Research suggests that children start developing number sense in early infancy (Feigenson, Dehaene, and Spelke 2004). Much of what preschool children know about number is closely related to and depends on their understanding and mastery of counting (*Adding It Up* 2001). Counting builds a foundation for children's future understanding of mathematics, and this basic skill becomes the reference point as children learn to manipulate larger quantities in the future.

Children's understanding of numbers is initially qualitative, as they gain an understanding of "number-ness" (e.g., three-ness, four-ness) with small quantities, using subitizing: visually knowing "how many" are in a set without actually counting them (Clements 2004a; Fuson 1988, 1992a). Counting is a natural activity for young children as their everyday contexts often involve numbers and quantities, although it requires them to have a sophisticated set of skills based on many experiences to be able to count accurately.

Literature suggests that the three major basic building blocks for counting are learning of (1) the sequence of number words, (2) one-to-one correspondence, and (3) cardinality (knowing that the last number assigned to the last object counted gives the total number in the set) (*Adding It Up* 2001; Becker 1989; Clements 2004a; Fuson 1988, 1992a, 1992b; Hiebert and others 1997; Sophian 1988). Children are likely to experience the aspects of counting at different times and in different contexts. As they gain more experience, they start to connect and coordinate these individual concepts and develop skill in counting with fluency. The specific ways in which these different aspects of counting develop depend largely on individual children and their experiences. Research, however, is in agreement that very young children (ages up to three) may be able to handle small quantities first (groups of two to three), and as they grow older, they are more likely to be able to manage larger sets (by age five, groups of 10). Cardinality is typically developed between the ages of three and four years (Fuson 1988). The preschool years are a critical time for children to master the art of counting small numbers of objects.

Young children's understanding of quantities and numbers is largely related to counting, as noted in the previous section. Another important factor in children's development of number sense is early experience with number operations (*Adding It Up* 2001; Clements 2004a; Hiebert and others 1997; *Principles and Standards for School Mathematics* 2000). Research shows that counting and number operations are related and that children as young as three years are able to understand simple visual number patterns that involve number operations such as, "two fingers and two fingers make four" (Fuson 1988, 1992a). When children enter elementary schools, much of their engagement with mathematics will be devoted to learning how quantitative and logical relationships work in the world, and number operations hold a key to such learning (*Adding It Up* 2001; *Principles and Standards for School Mathematics* 2000). Although standard mathematical and abstract symbols (e.g., +, =) are absent from those early experiences with math operations, informal and early mathematics experience becomes the foundation for children's later learning in this area. Children generally use a diverse range of strategies to make sense of mathematical situations around them, and this diversity of thinking usually becomes a feature of their subsequent mathematical development (*Adding It Up* 2001).

Young children initially understand a quantity as an aggregate of single units (Fuson 1988, 1992a, 1992b; Carpenter and Moser 1988; Hiebert and others 1997; Geary 1994). Thus, when asked to combine two sets of objects, they count the two different sets starting from "one" to determine the answer (the counting-all strategy); therefore, the development of number operations is closely related to the way they learn to count. As children gain experiences, they gradually develop more sophisticated methods by abstracting the quantity of one of the two groups (one of the addends) and starting to count on (or count up, in subtraction) from that number. Children eventually become adept at decomposing numbers into smaller chunks for the purpose of adding and subtracting, although this method is usually not formally taught in U.S. classrooms (Fuson, 1992a). Nevertheless, preschool children's first experiences with the concept of decomposition of a number into smaller groups of numbers is the beginning of an important development in mathematical reasoning. Learning the concept that groups or chunks of numbers make up larger numbers supports the understanding of arithmetic operations. For example, children's emerging understanding of different ways the number 10 can be decomposed into groups (e.g., 5 + 5, 4 + 6) contributes to their future learning of multidigit addition and subtraction (i.e., the operation of making 10 and moving it to the next position to the left of a multidigit number).

Algebra and Functions

During the preschool years children develop beginning algebraic concepts as they sort and classify objects, observe patterns in their environment, and begin to predict what comes next based on a recognized pattern. Sorting items, classifying them, and working with patterns help children to bring order, organization, and predictability to their world. Classification and the analysis of patterns provide a foundation for algebraic thinking as children develop the ability to recognize relationships, form generalizations, and see the connection between common underlying structures (*Principles and Standards for School Mathematics* 2000; Clements 2004a).

Classification is the systematic arrangement of objects into groups according to established criteria and involves sorting, grouping, and categorizing. Classification is at the heart of identifying what is *invariant* across groups of mathematical objects or mathematical processes. Clements (2004a) suggests that analyzing, comparing, and classifying objects help create new knowledge of objects and their relationships; in *Developmental Guidelines for Geometry,* he recommends a classification activity in which four-year-olds match shapes to identify congruent and noncongruent two-dimensional shapes. Certainly, identifying triangles from within a set of figures that include examples and nonexamples of triangles is essentially a classification exercise. But classification should not be reserved solely for work with shapes; rather, it should be included in young children's mathematical activities as it also facilitates work with patterns and data analysis. The developmental continuum for data analysis starts with classification and counting and evolves into data representation (e.g., graphing).

Seo and Ginsburg (2004) were interested in how frequently four- and five-year-olds engaged in mathematical activities during play. Interestingly enough, after studying 90 of these children the researchers report that classification activities were the least frequently occurring of the mathematical activities observed. Only 2 percent of the mathematical activities observed could be categorized as classification activities.

Patterns help children learn to find order, cohesion, and predictability in seemingly disorganized situations. The recognition and analysis of patterns clearly provide a foundation for the development of algebraic thinking (Clements 2004a). Identifying and extending patterns are important preschool activities. For example, Ginsburg, Inoue, and Seo (1999) report that the detection, prediction, and creation of patterns with shapes are the most frequent mathematical activities in preschool. However, compared with counting, little is known about young children's knowledge of patterns.

Patterns involve replication, completion, prediction, extension, and description or generalization (Greenes 1999). In preschool years, young children gradually develop the concept of patterns that includes recognizing a pattern, describing a pattern, creating a pattern, and extending a pattern. To understand a pattern, children should be able to identify similarities and differences among elements of a pattern, note the number of elements in the repeatable group, identify when the first group of elements begins to replicate itself, and make predictions about the order of elements based on given information.

Klein and Starkey (2004) report that young children experience difficulty at the beginning of the year with a fundamental property of repeating patterns: identifying the core unit of the pattern. However, experiences can have a positive impact on young children's knowledge of duplication and extension of patterns (Klein and Starkey 2004; Starkey, Klein, and Wakeley 2004).

In a study about the kinds of mathematical activities in which young children engage during play, Seo and Ginsburg (2004) found that four- and five-year-old children most often engage in "pattern and shape" activities, which the authors describe as ". . . identifying or creating patterns or shapes or exploring geometric properties and relationships. For example, Jennie makes a bead necklace, putting plastic beads into a string one by one. She uses only yellow and red beads for her necklace and makes a yellow-red color pattern" (Seo and Ginsburg 2004, 94).

These researchers provide some evidence that young children, when engaged in play, do generate their own repeating patterns. In preschool settings, teachers can encourage children to share their patterns created with objects, bodies, and sounds in relation to music, art, and movement (Smith 2001). Although the cited work is invaluable to the education of young children and the development of preschool learning foundations, much research remains to be done.

The developmental trajectory of patterns has been characterized as evolving from three-year-old children's ability to identify repeating pattern to four-year-old children's ability to engage in pattern duplication and pattern extension (Klein and Starkey 2004). The perception of the initial unit plays a fundamental role in both the duplication and extension of patterns.

Measurement

Measurement is defined as a mathematical process that involves assigning numbers to a set of continuous quantities (Clements and Stephen 2004). Technically, measurement is a number that indicates a comparison between the attribute of the object being measured and the same attribute of a given unit of measure. To understand the concept of measurement, children must be able to decide on the attribute of objects to measure, select the units to measure the attribute, and use measuring skills and tools to compare the units (Clements 2004a; Van de Walle 2001). To accomplish this task, children should understand the different units that are assigned to physical quantities such as length, height, weight, volume, and nonphysical quantities such as time, and temperature (Smith 2001).

Measurement is one of the main real-world applications of mathematics. Shaw and Blake (1998) note that in children's mathematics curricula, measurement is an integration of number operation and geometry in everyday mathematical experiences. A typical developmental trajectory involves children first learning to use words that represent quantities or magnitude of a certain attribute. Then, children begin to demonstrate an ability to compare two objects directly and recognize equality or inequality. For example, they may compare two objects to determine which is longer or heavier. After comparing two items, children develop the ability to compare three or more objects and to order them by size (e.g. from shortest to longest) or by other attributes. Finally, children learn to measure, connecting numbers to attributes of objects, such as length, weight, amount, and area (Clements 2004a; Ginsburg, Inoue, and Seo 1999).

This theoretical sequence establishes the basis for the measurement strand. Children's familiarity with the language required to describe measurement relationships—such as longer, taller, shorter, the same length, holds less, holds the same amount—is an important foundation for the concept of measurement (Greenes 1999) that should be directly addressed in preschool and, thus, is incorporated as part of the mathematics foundations for children at around 48 months of age. Young preschoolers learn to use words that describe measurement relationships as they compare two objects directly to determine equality or inequality, and as they order three or more objects by size. Older preschool children begin to make progress in reasoning about measuring quantities with less dependence on perceptual cues (Clements 2004a, Clements and Stephen 2004). Children start to compare the length of objects, indirectly, using transitive reasoning, and to measure the length of objects often by using nonstandard units. They develop the ability to think of the length of a small unit (i.e., a block) as part of the length of the object being measured and to place the smaller unit repeatedly along the length of the larger object.

Geometry

Geometry is the study of space and shape (Clements 1999). Geometry and spatial reasoning offer a way to describe, interpret, and imagine the world. They also provide an important tool for the study of mathematics and science. The research literature shows that young children bring to kindergarten a great deal of knowledge about shapes. This finding is important because teachers and curriculum writers seem to underestimate the knowledge about geometric figures that students bring to school. This underestimation and teachers' lack of confidence in their own geometry knowledge usually result in teachers' minimizing the time dedicated to teaching geometry concepts to children (Clements 2004a; Lehrer, Jenkins, and Osana 1998).

The literature recommends that young children be given the opportunity to work with many varied examples of a particular shape and many "nonexamples" of a particular shape (Clements 2004a). For example, children need to experience examples of triangles that are not just isosceles triangles. They need to experience triangles that are skewed—that is, a triangle where the "top" is not "in the middle," as in an isosceles triangle. They need also to experience triangles with a varying *aspect ratio*—the ratio of height to base. Without the opportunity to experience a wide range of triangles, children may come to "expect" triangles to have an aspect ratio that is close to 1 and, consequently, often reject appropriate examples of triangles because they are too "pointy" or too "flat." In addition, children need to experience nonexamples of triangles so that they can develop a robust and explicit sense of the properties of a triangle.

In 1959, Van Hiele developed a hierarchy of ways of understanding spatial ideas (Van Hiele 1986). The hierarchy consists of the following levels: 1—Visualization, 2—Analysis, 3—Abstraction, 4—Deduction, and 5—Rigor. Van Hiele's theory has become the most influential factor in geometry curricula. Recently, researchers have suggested a level of geometric thinking that exists before the visual level: a "precognition" level in which children cannot yet reliably identify circles, triangles, and squares (Clements 2004b; Clements and others 1999).

Shape knowledge involves not only recognition and naming but also an understanding of shape characteristics and properties. One way in which children demonstrate this understanding is through their ability to put together shapes into new shapes (Clements

2004a). The developmental trajectory for the composition of geometric figures evolves as children begin to use shapes individually to represent objects, progress to covering an outline with shapes, and eventually be able to combine shapes without an outline and make shape units (i.e., smaller shapes that make up a larger shape that is itself a part of a larger picture) (Clements 2004a; Clements and Sarama 2000).

Developing a sense of space is as important as developing spatial sense. Spatial sense allows people to get around in the world and know the relative positions of artifacts in the physical environment (Smith 2001). Spatial reasoning involves location, direction, distance, and identification of objects (Clements 1999). Very young children do develop an initial spatial sense to get around in the world. For example, young preschoolers learn to navigate their way around their school and classroom, and this ability suggests that they have created a mental map of those places. In the beginning stages of spatial reasoning, children use their own position as a point of reference for locating positions and orientations of objects in space, such as in/out and above/below. Then, children develop the ability to relate positions of two objects external to themselves or in themselves such as in front/in back, forward/backward, near/far, close to/far from (Greenes 1999). There is evidence that even preschool children develop mapping skills. They can build maps using familiar objects and as they get older, build imagery maps in familiar classroom settings (Blaut and Stea 1974; Gouteux and Spelke 2004; Rieser, Garing, and Young 1994).

Children's growth in understanding and knowledge about shape and space is thought to develop through education and experience rather than merely through maturational factors. Therefore, it is important not only to create a foundation for addressing this mathematics area, but also to encourage preschool programs to provide children with plenty of rich and varied opportunities to engage with various aspects of geometry. Engagement should be done in such a way that it grounds young children's experiences with shapes in action. As a result, the preschool foundations tend to de-emphasize the "naming" of shapes in this foundation; rather, they focus on children's ability to identify shapes, whether verbally or nonverbally.

Mathematical Reasoning

Mathematical proficiency entails strategic competence, adaptive reasoning, conceptual understanding, productive disposition, and procedural fluency (*Adding It Up* 2001). Each of these competencies sets the foundation for what is often called problem solving or mathematical reasoning.

Most preschool children by at least three years of age show that they can solve problems involving simple addition and subtraction, often by modeling with real objects or thinking about sets of objects. In a study by Huttenlocher, Jordan, and Levine (1994), preschoolers were presented with a set of objects of a given size that were then hidden in a box, followed by another set of objects that were also placed in the box. The children were asked to produce a set of objects corresponding to the total number of objects contained in the box. The majority of three-year-olds were able to solve these types of problems when they involved adding or subtracting a single item, but their performance decreased rapidly as the size of the second set increased.

Preschoolers demonstrate the conceptual understanding and procedural fluency necessary for them to solve simple word problems (Fuson 1992b). Simple word problems are thought to be easier for preschool children to solve than number problems that are not cast in a context (Carpenter and others 1993). All ages of problem-solvers are influenced by the context of the problem and tend to perform better with more contextual information (Wason and Johnson-Laird 1972; Shannon 1999). However, preschool children tend to be more heavily influenced by the context of the problem than do older children and adults, thus limiting their ability to solve number problems that are not presented in context.

Alexander, White, and Daugherty (1997) propose three conditions for reasoning in young children: the children must have a sufficient knowledge base, the task must be understandable and motivating, and the context of the task must be familiar and comfortable to the problem-solver. These conditions probably apply to all ages of problem-solvers (Wason and Johnson-Laird 1972; Shannon 1999).

Researchers indicate that four- and five-year-olds engage in advanced mathematical explorations spontaneously in their play (Ginsburg, Inoue, and Seo 1999; Seo and Ginsburg 2004). In their everyday activities, young children spontaneously engage in a variety of mathematical explorations and applications such as pattern analysis, change and transformation, comparison of magnitude, and estimations. Any logical thinking that children exhibit to solve real-life problems could potentially be considered beginning mathematical reasoning. For example, children distributing the same (or almost same) amount of snack to classmates or using strategies to solve immediate situations in play are situations in which children begin to demonstrate their ability to solve mathematical problems. Thus, it is crucial for teachers to be attuned to the fact that mathematical reasoning happens all the time in children's lives, and teachers would do well to use those occasions to nurture children's mathematical thinking skills. The examples illustrate the authentic problems that occur in preschoolers' everyday activities and all the different skills involved in mathematical reasoning and problem solving.

Glossary

attribute. A property or characteristic of an object or a person; attributes such as size, color, or shape would be apparent to a preschool child and would be used in grouping or sorting

cardinality. The concept that the number name applied to the last object counted represents the total number of objects in the group (the quantity of objects counted)

classification. The sorting, grouping, or categorizing of objects according to established criteria

one-to-one correspondence. One and only one number word is used for each object in the array of objects being counted

simple repeating pattern. A pattern with two repeating elements: A-B-A-B, A-A-B-B, A-B-B-A-B-B

subitize. The ability to quickly and accurately determine the quantity of objects in a small group (of up to five objects) without actually counting the objects

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Mathematics

Number Sense

At around 48 months of age	*At around 60 months of age*
1.0 Children begin to understand numbers and quantities in their everyday environment.	**1.0 Children expand their understanding of numbers and quantities in their everyday environment.**
1.1 Recite numbers in order to ten with increasing accuracy.	1.1 Recite numbers in order to twenty with increasing accuracy.
1.2 Begin to recognize and name a few written numerals.	1.2 Recognize and know the name of some written numerals.
1.3 Identify, without counting, the number of objects in a collection of up to three objects (i.e., subitize).	1.3 Identify, without counting, the number of objects in a collection of up to four objects (i.e., subitize).
1.4 Count up to five objects, using one-to-one correspondence (one object for each number word) with increasing accuracy.	1.4 Count up to ten objects, using one-to-one correspondence (one object for each number word) with increasing accuracy.
1.5 Use the number name of the last object counted to answer the question, “How many . . . ?”	1.5 Understand, when counting, that the number name of the last object counted represents the total number of objects in the group (i.e., cardinality).
2.0 Children begin to understand number relationships and operations in their everyday environment.	**2.0 Children expand their understanding of number relationships and operations in their everyday environment.**
2.1 Compare visually (with or without counting) two groups of objects that are obviously equal or nonequal and communicate, “more” or “same.”	2.1 Compare, by counting or matching, two groups of up to five objects and communicate, “more,” “same as,” or “fewer” (or “less”).
2.2 Understand that adding to (or taking away) one or more objects from a group will increase (or decrease) the number of objects in the group.	2.2 Understand that adding one or taking away one changes the number in a small group of objects by exactly one.

At around 48 months of age	At around 60 months of age
2.3 Understand that putting two groups of objects together will make a bigger group.	2.3 Understand that putting two groups of objects together will make a bigger group and that a group of objects can be taken apart into smaller groups.
2.4 Solve simple addition and subtraction problems nonverbally (and often verbally) with a very small number of objects (sums up to 4 or 5).	2.4 Solve simple addition and subtraction problems with a small number of objects (sums up to 10), usually by counting.

Algebra and Functions
(Classification and Patterning)

At around 48 months of age	At around 60 months of age
1.0 Children begin to sort and classify objects in their everyday environment.	**1.0 Children expand their understanding of sorting and classifying objects in their everyday environment.**
1.1 Sort and classify objects by one attribute into two or more groups, with increasing accuracy.	1.1 Sort and classify objects by one or more attributes, into two or more groups, with increasing accuracy (e.g., may sort first by one attribute and then by another attribute).
2.0 Children begin to recognize simple, repeating patterns.	**2.0 Children expand their understanding of simple, repeating patterns.**
2.1 Begin to identify or recognize a simple repeating pattern.	2.1 Recognize and duplicate simple repeating patterns.
2.2 Attempt to create a simple repeating pattern or participate in making one.	2.2 Begin to extend and create simple repeating patterns.

Measurement

At around 48 months of age	*At around 60 months of age*
1.0 Children begin to compare and order objects.	**1.0 Children expand their understanding of comparing, ordering, and measuring objects.**
1.1 Demonstrate awareness that objects can be compared by length, weight, or capacity, by noting gross differences, using words such as *bigger, longer, heavier,* or *taller,* or by placing objects side by side to compare length.	1.1 Compare two objects by length, weight, or capacity directly (e.g., putting objects side by side) or indirectly (e.g., using a third object).
1.2 Order three objects by size.	1.2 Order four or more objects by size.
	1.3 Measure length using multiple duplicates of the same-size concrete units laid end to end.

Geometry

At around 48 months of age	*At around 60 months of age*
1.0 Children begin to identify and use common shapes in their everyday environment.	**1.0 Children identify and use a variety of shapes in their everyday environment.**
1.1 Identify simple two-dimensional shapes, such as a circle and square.	1.1 Identify, describe, and construct a variety of different shapes, including variations of a circle, triangle, rectangle, square, and other shapes.
1.2 Use individual shapes to represent different elements of a picture or design.	1.2 Combine different shapes to create a picture or design.
2.0 Children begin to understand positions in space.	**2.0 Children expand their understanding of positions in space.**
2.1 Identify positions of objects and people in space, such as in/on/under, up/down, and inside/outside.	2.1 Identify positions of objects and people in space, including in/on/under, up/down, inside/outside, beside/between, and in front/behind.

Mathematical Reasoning

At around 48 months of age	*At around 60 months of age*
1.0 Children use mathematical thinking to solve problems that arise in their everyday environment.	**1.0 Children expand the use of mathematical thinking to solve problems that arise in their everyday environment.**
1.1 Begin to apply simple mathematical strategies to solve problems in their environment.	1.1 Identify and apply a variety of mathematical strategies to solve problems in their environment.

Common Core State Standards

for Mathematics

for California Public Schools Kindergarten Through Grade Twelve

COMMON CORE
STATE STANDARDS
CALIFORNIA

Adopted by the California State Board of Education
August 2010
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Prepublication Version

Contents

A Message from the State Board of Education and the State Superintendent of Public Instruction

A Message from the State Board of Education and State Superintendent of Public Instruction:

The Common Core State Standards for Mathematics (CCSSM) reflect the importance of focus, coherence, and rigor as the guiding principles for mathematics instruction and learning. California's implementation of the CCSSM demonstrates a commitment to providing a world-class education for all students that supports college and career readiness and the knowledge and skills necessary to fully participate in the 21st century global economy.

The CCSSM build on California's standards-based educational system in which curriculum, instruction, professional learning, assessment, and accountability are aligned to support student attainment of the standards. The CCSSM incorporate current research and input from other education stakeholders – including other state departments of education, scholars, professional organizations, teachers and other educators, parents, and students. A number of California-specific additions to the standards (identified in bolded text and followed by the "CA" state acronym) were incorporated in an effort to retain the consistency and precision of our past standards. The CCSSM are internationally benchmarked, research-based, and unequivocally rigorous.

The standards call for learning mathematical content in the context of real-world situations, using mathematics to solve problems, and developing "habits of mind" that foster mastery of mathematics content as well as mathematical understanding. The standards for kindergarten through grade eight prepare students for higher mathematics. The standards for higher mathematics reflect the knowledge and skills that are necessary to prepare students for college and career and productive citizenship.

Implementation of the CCSSM will take time and effort, but it also provides a new and exciting opportunity to ensure that California' students are held to the same high expectations in mathematics as their national and global peers. While California educators have implemented standards before, the CCSSM require not only rigorous curriculum and instruction but also conceptual understanding, procedural skill and fluency, and the ability to apply mathematics. In short, the standards call for meeting the challenges of the 21^{st} century through innovation.

DR. MICHAEL KIRST, President
California State Board of Education

TOM TORLAKSON
State Superintendent of Public Instruction

Introduction

All students need a high-quality mathematics program designed to prepare them to graduate from high school ready for college and careers. In support of this goal, California adopted the Common Core State Standards for Mathematics with California Additions (CCSSM) in June 2010, replacing the 1997 statewide mathematics academic standards. As part of the modification of the CCSSM in January 2013, the California State Board of Education also approved higher mathematics standards organized into model courses.

The CCSSM are designed to be robust, linked within and across grades, and relevant to the real world, reflecting the knowledge and skills that our young people need for success in college and careers. With California's students fully prepared for the future, our students will be positioned to compete successfully in the global economy

The development of these standards began as a voluntary, state-led effort coordinated by the Council of Chief State School Officers (CCSSO) and the National Governors Association Center for Best Practices (NGA) committed to developing a set of standards that would help prepare students for success in career and college. The CCSSM are based on evidence of the skills and knowledge needed for college and career readiness and an expectation that students be able to both know and do mathematics by solving a range of problems and engaging in key mathematical practices.

The development of these standards was informed by international benchmarking and began with research-based learning progressions detailing what is known about how students' mathematical knowledge, skills, and understanding develop over time. The progression from kindergarten standards to standards for higher mathematics exemplifies the three principles of focus, coherence, and rigor that are the basis for the CCSSM.

The first principle, focus, implies that instruction should focus deeply on only those concepts that are emphasized in the standards so that students can gain strong foundational conceptual understanding, a high degree of procedural skill and fluency, and the ability to apply the mathematics they know to solve problems inside and outside the mathematics classroom. Coherence arises from mathematical connections. Some of the connections in the standards knit topics together at a single grade level. Most connections are vertical, as the standards support a progression of increasing knowledge, skill, and sophistication across the grades. Finally, rigor requires that conceptual understanding, procedural skill and fluency, and application be approached with equal intensity.

Two Types of Standards

The CCSSM include two types of standards: Eight Mathematical Practice Standards (the same at each grade level) and Mathematical Content Standards (different at each grade level). Together these standards address both “habits of mind” that students should develop to foster mathematical understanding and expertise and skills and knowledge – what students need to know and be able to do. The mathematical content standards were built on progressions of topics across a number of grade levels, informed both by research on children's cognitive development and by the logical structure of mathematics.

The Standards for Mathematical Practice (MP) are the same at each grade level, with the exception of an additional practice standard included in the California CCSSM for higher mathematics only: **MP3.1: Students build proofs by induction and proofs by contradiction. CA** This standard can be seen as an extension of Mathematical Practice 3, in which students construct viable arguments and critique the reasoning of others. Ideally, several MP standards will be evident in each lesson as they interact and overlap with each other. The MP standards are not a checklist; they are the basis for mathematics instruction and learning. Structuring the MP standards can help educators recognize opportunities for students to engage with mathematics in grade-appropriate ways. The eight MP standards can be grouped into the four categories as illustrated in the following chart.

Structuring the Standards for Mathematical Practice[1]

1. Make sense of problems and persevere in solving them. 6. Attend to precision.	2. Reason abstractly and quantitatively. 3. Construct viable arguments and critique the reasoning of others.	Reasoning and explaining
	4. Model with mathematics. 5. Use appropriate tools strategically.	Modeling and using tools
	7. Look for and make use of structure. 8. Look for and express regularity in repeated reasoning.	Seeing structure and generalizing

The CCSSM call for mathematical practices and mathematical content to be connected as students engage in mathematical tasks. These connections are essential to support the development of

[1] McCallum, Bill. 2011. *Structuring the Mathematical Practices*. http://commoncoretools.me/wp-content/uploads/2011/03/practices.pdf (accessed April 1, 2013).

students' broader mathematical understanding – students who lack understanding of a topic may rely on procedures too heavily. The MP standards must be taught as carefully and practiced as intentionally as the Mathematical Content Standards. Neither should be isolated from the other; effective mathematics instruction occurs when these two halves of the CCSSM come together in a powerful whole.

How to Read the Standards

Kindergarten–Grade 8

In kindergarten through grade eight the CCSSM are organized by grade level and then by domains (clusters of standards that address "big ideas" and support connections of topics across the grades), clusters (groups of related standards inside domains) and finally by the standards (what students should understand and be able to do). The standards do not dictate curriculum or pedagogy. For example, just because Topic A appears before Topic B in the standards for a given grade, it does not mean that Topic A must be taught before Topic B.

The code for each standard begins with the grade level, followed by the domain code, and the standard number. For example, 3.NBT 2. would be the second standard in the Number and Operations in Base Ten domain of the standards for grade three.

Domain

Number and Operations in Base Ten 3.NBT

Use place value understanding and properties of operations to perform multi-digit arithmetic.

1. Use place value understanding to round whole numbers to the nearest 10 or 100.
2. Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.
3. Multiply one-digit whole numbers by multiples of 10 in the range 10–90 (e.g., 9 × 80, 5 × 60) using strategies based on place value and properties of operations.

Standard

Cluster

Higher Mathematics

In California, the CCSSM for higher mathematics are organized into both model courses and conceptual categories. The higher mathematics courses adopted by the State Board of Education in January 2013 are based on the guidance provided in Appendix A published by the Common Core

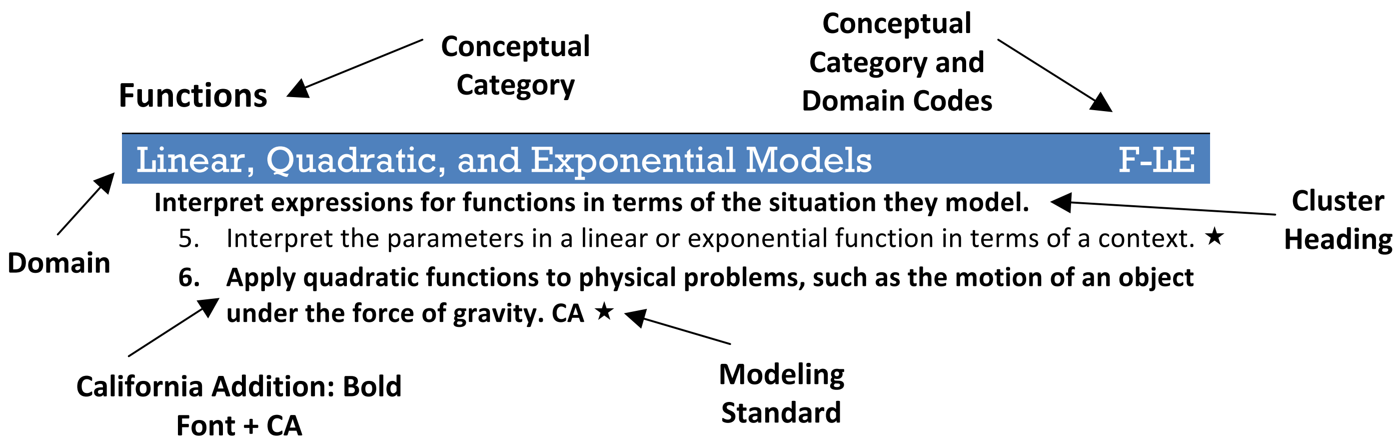
State Standards Initiative.[2] The model courses for higher mathematics are organized into two pathways: traditional and integrated. The traditional pathway consists of the higher mathematics standards organized along more traditional lines into Algebra I, Geometry and Algebra II courses. The integrated pathway consists of the courses Mathematics I, II and III. The integrated pathway presents higher mathematics as a connected subject, in that each course contains standards from all six of the conceptual categories. In addition, two advanced higher mathematics courses were retained from the 1997 mathematics standards, Advanced Placement Probability and Statistics and Calculus.

The standards for higher mathematics are also listed in conceptual categories:

- Number and Quantity
- Algebra
- Functions
- Modeling
- Geometry
- Statistics and Probability

The conceptual categories portray a coherent view of higher mathematics based on the realization that students' work on a broad topic, such as functions, crosses a number of traditional course boundaries. As local school districts develop a full range of courses and curriculum in higher mathematics, the organization of standards by conceptual categories offers a starting point for discussing course content.

The code for each higher mathematics standard begins with the identifier for the conceptual category code (N, A, F, G, S), followed by the domain code, and the standard number. For example, F-LE.5 would be the fifth standard in the Linear, Quadratic, and Exponential Models domain of the conceptual category of Functions.



[2] Appendix A provides guidance to the field on developing higher mathematics courses. This appendix is available on the Common Core State Standards Initiative Web site at: http://www.corestandards.org/Math.

The star symbol (★) following the standard indicates those that are also Modeling standards. Modeling is best interpreted not as a collection of isolated topics but in relation to other standards. Making mathematical models is a MP standard and specific modeling standards appear throughout the higher mathematics standards indicated by a star symbol (★). Additional mathematics that students should learn in order to take advanced courses such as calculus, advanced statistics, or discrete mathematics is indicated by a (+) symbol. Standards with a (+) symbol may appear in courses intended for all students.

Mathematics | Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

1 Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2 Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3 Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments. **Students build proofs by induction and proofs by contradiction. CA 3.1** (for higher mathematics only).

4 Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5 Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6 Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7 Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 x 8 equals the well-remembered 7 x 5 + 7 x 3, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2 x 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 – 3(x – y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

8 Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y – 2)/(x – 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x – 1)(x + 1)$, $(x – 1)(x^2 + x + 1)$, and $(x – 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word “understand” are often especially good opportunities to connect the practices to the content. Students who lack understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a lack of understanding effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential “points of intersection” between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.

K–8 Standards

Mathematics | Kindergarten

In Kindergarten, instructional time should focus on two critical areas:(1) representing, relating, and operating on whole numbers, initially with sets of objects; and (2) describing shapes and space. More learning time in Kindergarten should be devoted to number than to other topics.

(1) Students use numbers, including written numerals, to represent quantities and to solve quantitative problems, such as counting objects in a set; counting out a given number of objects; comparing sets or numerals; and modeling simple joining and separating situations with sets of objects, or eventually with equations such as $5 + 2 = 7$ and $7 – 2 = 5$. (Kindergarten students should see addition and subtraction equations, and student writing of equations in kindergarten is encouraged, but it is not required.) Students choose, combine, and apply effective strategies for answering quantitative questions, including quickly recognizing the cardinalities of small sets of objects, counting and producing sets of given sizes, counting the number of objects in combined sets, or counting the number of objects that remain in a set after some are taken away.

(2) Students describe their physical world using geometric ideas (e.g., shape, orientation, spatial relations) and vocabulary. They identify, name, and describe basic two-dimensional shapes, such as squares, triangles, circles, rectangles, and hexagons, presented in a variety of ways (e.g., with different sizes and orientations), as well as three-dimensional shapes such as cubes, cones, cylinders, and spheres. They use basic shapes and spatial reasoning to model objects in their environment and to construct more complex shapes.

K

Grade K Overview

Counting and Cardinality

- Know number names and the count sequence.
- Count to tell the number of objects.
- Compare numbers.

Operations and Algebraic Thinking

- Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.

Number and Operations in Base Ten

- Work with numbers 11–19 to gain foundations for place value.

Measurement and Data

- Describe and compare measurable attributes.
- Classify objects and count the number of objects in categories.

Geometry

- Identify and describe shapes.
- Analyze, compare, create, and compose shapes.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

K Grade K

Counting and Cardinality K.CC

Know number names and the count sequence.

1. Count to 100 by ones and by tens.
2. Count forward beginning from a given number within the known sequence (instead of having to begin at 1).
3. Write numbers from 0 to 20. Represent a number of objects with a written numeral 0-20 (with 0 representing a count of no objects).

Count to tell the number of objects.

4. Understand the relationship between numbers and quantities; connect counting to cardinality.
   a. When counting objects, say the number names in the standard order, pairing each object with one and only one number name and each number name with one and only one object.
   b. Understand that the last number name said tells the number of objects counted. The number of objects is the same regardless of their arrangement or the order in which they were counted.
   c. Understand that each successive number name refers to a quantity that is one larger.
5. Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects.

Compare numbers.

6. Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies.[1]
7. Compare two numbers between 1 and 10 presented as written numerals.

Operations and Algebraic Thinking K.OA

Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.

1. Represent addition and subtraction with objects, fingers, mental images, drawings[2], sounds (e.g., claps), acting out situations, verbal explanations, expressions, or equations.
2. Solve addition and subtraction word problems, and add and subtract within 10, e.g., by using objects or drawings to represent the problem.
3. Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., 5 = 2 + 3 and 5 = 4 + 1).
4. For any number from 1 to 9, find the number that makes 10 when added to the given number, e.g., by using objects or drawings, and record the answer with a drawing or equation.
5. Fluently add and subtract within 5.

[1] Includes groups with up to ten objects.
[2] Drawings need not show details, but should show the mathematics in the problem.
(This applies wherever drawings are mentioned in the Standards)

K Grade K

Number and Operations in Base Ten K.NBT

Work with numbers 11–19 to gain foundations for place value.

1. Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (e.g., 18 = 10 + 8); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones.

Measurement and Data K.MD

Describe and compare measurable attributes.

1. Describe measurable attributes of objects, such as length or weight. Describe several measurable attributes of a single object.
2. Directly compare two objects with a measurable attribute in common, to see which object has "more of"/"less of" the attribute, and describe the difference. *For example, directly compare the heights of two children and describe one child as taller/shorter.*

Classify objects and count the number of objects in each category.

3. Classify objects into given categories; count the numbers of objects in each category and sort the categories by count.[3]

Geometry K.G

Identify and describe shapes (squares, circles, triangles, rectangles, hexagons, cubes, cones, cylinders, and spheres).

1. Describe objects in the environment using names of shapes, and describe the relative positions of these objects using terms such as *above*, *below*, *beside*, *in front of*, *behind*, and *next to*.
2. Correctly name shapes regardless of their orientations or overall size.
3. Identify shapes as two-dimensional (lying in a plane, "flat") or three-dimensional ("solid").

Analyze, compare, create, and compose shapes.

4. Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts (e.g., number of sides and vertices/"corners") and other attributes (e.g., having sides of equal length).
5. Model shapes in the world by building shapes from components (e.g., sticks and clay balls) and drawing shapes.
6. Compose simple shapes to form larger shapes. *For example, "Can you join these two triangles with full sides touching to make a rectangle?"*

[3] Limit category counts to be less than or equal to 10.

Mathematics | Grade 1

In Grade 1, instructional time should focus on four critical areas: (1) developing understanding of addition, subtraction, and strategies for addition and subtraction within 20; (2) developing understanding of whole number relationships and place value, including grouping in tens and ones; (3) developing understanding of linear measurement and measuring lengths as iterating length units; and (4) reasoning about attributes of, and composing and decomposing geometric shapes.

(1) Students develop strategies for adding and subtracting whole numbers based on their prior work with small numbers. They use a variety of models, including discrete objects and length-based models (e.g., cubes connected to form lengths), to model add-to, take-from, put-together, take-apart, and compare situations to develop meaning for the operations of addition and subtraction, and to develop strategies to solve arithmetic problems with these operations. Students understand connections between counting and addition and subtraction (e.g., adding two is the same as counting on two). They use properties of addition to add whole numbers and to create and use increasingly sophisticated strategies based on these properties (e.g., “making tens”) to solve addition and subtraction problems within 20. By comparing a variety of solution strategies, children build their understanding of the relationship between addition and subtraction.

(2) Students develop, discuss, and use efficient, accurate, and generalizable methods to add within 100 and subtract multiples of 10. They compare whole numbers (at least to 100) to develop understanding of and solve problems involving their relative sizes. They think of whole numbers between 10 and 100 in terms of tens and ones (especially recognizing the numbers 11 to 19 as composed of a ten and some ones). Through activities that build number sense, they understand the order of the counting numbers and their relative magnitudes.

(3) Students develop an understanding of the meaning and processes of measurement, including underlying concepts such as iterating (the mental activity of building up the length of an object with equal-sized units) and the transitivity principle for indirect measurement. [1]

(4) Students compose and decompose plane or solid figures (e.g., put two triangles together to make a quadrilateral) and build understanding of part-whole relationships as well as the properties of the original and composite shapes. As they combine shapes, they recognize them from different perspectives and orientations, describe their geometric attributes, and determine how they are alike and different, to develop the background for measurement and for initial understandings of properties such as congruence and symmetry.

[1] Students should apply the principle of transitivity of measurement to make indirect comparisons, but they need not use this technical term.

1 Grade 1 Overview

Operations and Algebraic Thinking

- Represent and solve problems involving addition and subtraction.
- Understand and apply properties of operations and the relationship between addition and subtraction.
- Add and subtract within 20.
- Work with addition and subtraction equations.

Number and Operations in Base Ten

- Extend the counting sequence.
- Understand place value.
- Use place value understanding and properties of operations to add and subtract.

Measurement and Data

- Measure lengths indirectly and by iterating length units.
- Tell and write time.
- Represent and interpret data.

Geometry

- Reason with shapes and their attributes.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

1 Grade 1

Operations and Algebraic Thinking 1.OA

Represent and solve problems involving addition and subtraction.

1. Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.[2]
2. Solve word problems that call for addition of three whole numbers whose sum is less than or equal to 20, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

Understand and apply properties of operations and the relationship between addition and subtraction.

3. Apply properties of operations as strategies to add and subtract.[3] *Examples: If 8 + 3 = 11 is known, then 3 + 8 = 11 is also known. (Commutative property of addition.) To add 2 + 6 + 4, the second two numbers can be added to make a ten, so 2 + 6 + 4 = 2 + 10 = 12. (Associative property of addition.)*
4. Understand subtraction as an unknown-addend problem. *For example, subtract 10 – 8 by finding the number that makes 10 when added to 8.*

Add and subtract within 20.

5. Relate counting to addition and subtraction (e.g., by counting on 2 to add 2).
6. Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., 8 + 6 = 8 + 2 + 4 = 10 + 4 = 14); decomposing a number leading to a ten (e.g., 13 – 4 = 13 – 3 – 1 = 10 – 1 = 9); using the relationship between addition and subtraction (e.g., knowing that 8 + 4 = 12, one knows 12 – 8 = 4); and creating equivalent but easier or known sums (e.g., adding 6 + 7 by creating the known equivalent 6 + 6 + 1 = 12 + 1 = 13).

Work with addition and subtraction equations.

7. Understand the meaning of the equal sign, and determine if equations involving addition and subtraction are true or false. *For example, which of the following equations are true and which are false? 6 = 6, 7 = 8 – 1, 5 + 2 = 2 + 5, 4 + 1 = 5 + 2.*
8. Determine the unknown whole number in an addition or subtraction equation relating three whole numbers. *For example, determine the unknown number that makes the equation true in each of the equations 8 + ? = 11, 5 = ▯ – 3, 6 + 6 = ▯.*

Number and Operations in Base Ten 1.NBT

Extend the counting sequence.

1. Count to 120, starting at any number less than 120. In this range, read and write numerals and represent a number of objects with a written numeral.

Understand place value.

2. Understand that the two digits of a two-digit number represent amounts of tens and ones. Understand the following as special cases:
    a. 10 can be thought of as a bundle of ten ones — called a "ten."

[2] See Glossary, Table 1.
[3] Students need not use formal terms for these properties.

1 Grade 1

   b. The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones.
   c. The numbers 10, 20, 30, 40, 50, 60, 70, 80, 90 refer to one, two, three, four, five, six, seven, eight, or nine tens (and 0 ones).
3. Compare two two-digit numbers based on meanings of the tens and ones digits, recording the results of comparisons with the symbols >, =, and <.

Use place value understanding and properties of operations to add and subtract.

4. Add within 100, including adding a two-digit number and a one-digit number, and adding a two-digit number and a multiple of 10, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. Understand that in adding two-digit numbers, one adds tens and tens, ones and ones; and sometimes it is necessary to compose a ten.
5. Given a two-digit number, mentally find 10 more or 10 less than the number, without having to count; explain the reasoning used.
6. Subtract multiples of 10 in the range 10-90 from multiples of 10 in the range 10-90 (positive or zero differences), using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

Measurement and Data 1.MD

Measure lengths indirectly and by iterating length units.

1. Order three objects by length; compare the lengths of two objects indirectly by using a third object.
2. Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps. *Limit to contexts where the object being measured is spanned by a whole number of length units with no gaps or overlaps.*

Tell and write time.

3. Tell and write time in hours and half-hours using analog and digital clocks.

Represent and interpret data.

4. Organize, represent, and interpret data with up to three categories; ask and answer questions about the total number of data points, how many in each category, and how many more or less are in one category than in another.

Geometry 1.G

Reason with shapes and their attributes.

1. Distinguish between defining attributes (e.g., triangles are closed and three-sided) versus non-defining attributes (e.g., color, orientation, overall size); build and draw shapes to possess defining attributes.
2. Compose two-dimensional shapes (rectangles, squares, trapezoids, triangles, half-circles, and quarter-circles) or three-dimensional shapes (cubes, right rectangular prisms, right circular cones, and right circular cylinders) to create a composite shape, and compose new shapes from the composite shape.[4]

[4] Students do not need to learn formal names such as “right rectangular prism.”

1 Grade 1

3. Partition circles and rectangles into two and four equal shares, describe the shares using the words *halves*, *fourths*, and *quarters*, and use the phrases *half of*, *fourth of*, and *quarter of*. Describe the whole as two of, or four of the shares. Understand for these examples that decomposing into more equal shares creates smaller shares.

Mathematics | Grade 2

In Grade 2, instructional time should focus on four critical areas: (1) extending understanding of base-ten notation; (2) building fluency with addition and subtraction; (3) using standard units of measure; and (4) describing and analyzing shapes.

(1) Students extend their understanding of the base-ten system. This includes ideas of counting in fives, tens, and multiples of hundreds, tens, and ones, as well as number relationships involving these units, including comparing. Students understand multi-digit numbers (up to 1000) written in base-ten notation, recognizing that the digits in each place represent amounts of thousands, hundreds, tens, or ones (e.g., 853 is 8 hundreds + 5 tens + 3 ones).

(2) Students use their understanding of addition to develop fluency with addition and subtraction within 100. They solve problems within 1000 by applying their understanding of models for addition and subtraction, and they develop, discuss, and use efficient, accurate, and generalizable methods to compute sums and differences of whole numbers in base-ten notation, using their understanding of place value and the properties of operations. They select and accurately apply methods that are appropriate for the context and the numbers involved to mentally calculate sums and differences for numbers with only tens or only hundreds.

(3) Students recognize the need for standard units of measure (centimeter and inch) and they use rulers and other measurement tools with the understanding that linear measure involves an iteration of units. They recognize that the smaller the unit, the more iterations they need to cover a given length.

(4) Students describe and analyze shapes by examining their sides and angles. Students investigate, describe, and reason about decomposing and combining shapes to make other shapes. Through building, drawing, and analyzing two- and three-dimensional shapes, students develop a foundation for understanding area, volume, congruence, similarity, and symmetry in later grades.

2 Grade 2 Overview

Operations and Algebraic Thinking

- Represent and solve problems involving addition and subtraction.
- Add and subtract within 20.
- Work with equal groups of objects to gain foundations for multiplication.

Number and Operations in Base Ten

- Understand place value.
- Use place value understanding and properties of operations to add and subtract.

Measurement and Data

- Measure and estimate lengths in standard units.
- Relate addition and subtraction to length.
- Work with time and money.
- Represent and interpret data.

Geometry

- Reason with shapes and their attributes.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

2 Grade 2

Operations and Algebraic Thinking 2.OA

Represent and solve problems involving addition and subtraction.

1. Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.[1]

Add and subtract within 20.

2. Fluently add and subtract within 20 using mental strategies.[2] By end of Grade 2, know from memory all sums of two one-digit numbers.

Work with equal groups of objects to gain foundations for multiplication.

3. Determine whether a group of objects (up to 20) has an odd or even number of members, e.g., by pairing objects or counting them by 2s; write an equation to express an even number as a sum of two equal addends.
4. Use addition to find the total number of objects arranged in rectangular arrays with up to 5 rows and up to 5 columns; write an equation to express the total as a sum of equal addends.

Number and Operations in Base Ten 2.NBT

Understand place value.

1. Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases:
   a. 100 can be thought of as a bundle of ten tens — called a “hundred.”
   b. The numbers 100, 200, 300, 400, 500, 600, 700, 800, 900 refer to one, two, three, four, five, six, seven, eight, or nine hundreds (and 0 tens and 0 ones).
2. Count within 1000; skip-count by **2s**, 5s, 10s, and 100s. **CA**
3. Read and write numbers to 1000 using base-ten numerals, number names, and expanded form.
4. Compare two three-digit numbers based on meanings of the hundreds, tens, and ones digits, using >, =, and < symbols to record the results of comparisons.

Use place value understanding and properties of operations to add and subtract.

5. Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.
6. Add up to four two-digit numbers using strategies based on place value and properties of operations.
7. Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.

7.1 Use estimation strategies to make reasonable estimates in problem solving. CA

8. Mentally add 10 or 100 to a given number 100–900, and mentally subtract 10 or 100 from a given number 100–900.

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[1] See Glossary, Table 1.
[2] See standard 1.OA.6 for a list of mental strategies.

2 Grade 2

9. Explain why addition and subtraction strategies work, using place value and the properties of operations.[3]

Measurement and Data 2.MD

Measure and estimate lengths in standard units.

1. Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.
2. Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.
3. Estimate lengths using units of inches, feet, centimeters, and meters.
4. Measure to determine how much longer one object is than another, expressing the length difference in terms of a standard length unit.

Relate addition and subtraction to length.

5. Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.
6. Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, ..., and represent whole-number sums and differences within 100 on a number line diagram.

Work with time and money.

7. Tell and write time from analog and digital clocks to the nearest five minutes, using a.m. and p.m. **Know relationships of time (e.g., minutes in an hour, days in a month, weeks in a year). CA**
8. Solve word problems involving dollar bills, quarters, dimes, nickels, and pennies, using $ and ¢ symbols appropriately. *Example: If you have 2 dimes and 3 pennies, how many cents do you have?*

Represent and interpret data.

9. Generate measurement data by measuring lengths of several objects to the nearest whole unit, or by making repeated measurements of the same object. Show the measurements by making a line plot, where the horizontal scale is marked off in whole-number units.
10. Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put-together, take-apart, and compare problems[4] using information presented in a bar graph.

Geometry 2.G

Reason with shapes and their attributes.

1. Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces.[5] Identify triangles, quadrilaterals, pentagons, hexagons, and cubes.
2. Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.

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[3] Explanations may be supported by drawings or objects.
[4] See Glossary, Table 1.
[5] Sizes are compared directly or visually, not compared by measuring.

2 Grade 2

3. Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words *halves*, *thirds*, *half of*, *a third of*, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

Mathematics | Grade 3

In Grade 3, instructional time should focus on four critical areas: (1) developing understanding of multiplication and division and strategies for multiplication and division within 100; (2) developing understanding of fractions, especially unit fractions (fractions with numerator 1); (3) developing understanding of the structure of rectangular arrays and of area; and (4) describing and analyzing two-dimensional shapes.

(1) Students develop an understanding of the meanings of multiplication and division of whole numbers through activities and problems involving equal-sized groups, arrays, and area models; multiplication is finding an unknown product, and division is finding an unknown factor in these situations. For equal-sized group situations, division can require finding the unknown number of groups or the unknown group size. Students use properties of operations to calculate products of whole numbers, using increasingly sophisticated strategies based on these properties to solve multiplication and division problems involving single-digit factors. By comparing a variety of solution strategies, students learn the relationship between multiplication and division.

(2) Students develop an understanding of fractions, beginning with unit fractions. Students view fractions in general as being built out of unit fractions, and they use fractions along with visual fraction models to represent parts of a whole. Students understand that the size of a fractional part is relative to the size of the whole. For example,1/2 of the paint in a small bucket could be less paint than 1/3 of the paint in a larger bucket, but 1/3 of a ribbon is longer than 1/5 of the same ribbon because when the ribbon is divided into 3 equal parts, the parts are longer than when the ribbon is divided into 5 equal parts. Students are able to use fractions to represent numbers equal to, less than, and greater than one. They solve problems that involve comparing fractions by using visual fraction models and strategies based on noticing equal numerators or denominators.

(3) Students recognize area as an attribute of two-dimensional regions. They measure the area of a shape by finding the total number of same-size units of area required to cover the shape without gaps or overlaps, a square with sides of unit length being the standard unit for measuring area. Students understand that rectangular arrays can be decomposed into identical rows or into identical columns. By decomposing rectangles into rectangular arrays of squares, students connect area to multiplication, and justify using multiplication to determine the area of a rectangle.

(4) Students describe, analyze, and compare properties of two-dimensional shapes. They compare and classify shapes by their sides and angles, and connect these with definitions of shapes. Students also relate their fraction work to geometry by expressing the area of part of a shape as a unit fraction of the whole.

3 Grade 3 Overview

Operations and Algebraic Thinking

- Represent and solve problems involving multiplication and division.
- Understand properties of multiplication and the relationship between multiplication and division.
- Multiply and divide within 100.
- Solve problems involving the four operations, and identify and explain patterns in arithmetic.

Number and Operations in Base Ten

- Use place value understanding and properties of operations to perform multi-digit arithmetic.

Number and Operations—Fractions

- Develop understanding of fractions as numbers.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Measurement and Data

- Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.
- Represent and interpret data.
- Geometric measurement: understand concepts of area and relate area to multiplication and to addition.
- Geometric measurement: recognize perimeter as an attribute of plane figures and distinguish between linear and area measures.

Geometry

- Reason with shapes and their attributes.

3 Grade 3

Operations and Algebraic Thinking 3.OA

Represent and solve problems involving multiplication and division.

1. Interpret products of whole numbers, e.g., interpret 5 × 7 as the total number of objects in 5 groups of 7 objects each. *For example, describe a context in which a total number of objects can be expressed as* 5 *× 7.*
2. Interpret whole-number quotients of whole numbers, e.g., interpret 56 ÷ 8 as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each. *For example, describe a context in which a number of shares or a number of groups can be expressed as 56÷8.*
3. Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.[1]
4. Determine the unknown whole number in a multiplication or division equation relating three whole numbers. For example, determine the unknown number that makes the equation true in each of the equations 8 × ? = 48, 5 = □ ÷ 3, 6 × 6 = ?.

Understand properties of multiplication and the relationship between multiplication and division.

5. Apply properties of operations as strategies to multiply and divide.[2] *Examples: If 6 × 4 = 24 is known, then 4 × 6 = 24 is also known. (Commutative property of multiplication.) 3 × 5 × 2 can be found by 3 × 5 = 15, then 15 × 2 = 30, or by 5 × 2 = 10, then 3 × 10 = 30. (Associative property of multiplication.) Knowing that 8 × 5 = 40 and 8 × 2 = 16, one can find 8 × 7 as 8 × (5 + 2) = (8 × 5) + (8 × 2) = 40 + 16 = 56. (Distributive property.)*
6. Understand division as an unknown-factor problem. *For example, find 32 ÷ 8 by finding the number that makes 32 when multiplied by 8.*

Multiply and divide within 100.

7. Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that 8 × 5 = 40, one knows 40 ÷ 5 = 8) or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers.

Solve problems involving the four operations, and identify and explain patterns in arithmetic.

8. Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.[3]
9. Identify arithmetic patterns (including patterns in the addition table or multiplication table), and explain them using properties of operations. *For example, observe that 4 times a number is always even, and explain why 4 times a number can be decomposed into two equal addends.*

[1] See Glossary, Table 2.
[2] Students need not use formal terms for these properties.
[3] This standard is limited to problems posed with whole numbers and having whole-number answers; students should know how to perform operations in the conventional order when there are no parentheses to specify a particular order (Order of Operations).

3 Grade 3

Number and Operations in Base Ten 3.NBT

Use place value understanding and properties of operations to perform multi-digit arithmetic.[4]

1. Use place value understanding to round whole numbers to the nearest 10 or 100.
2. Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.
3. Multiply one-digit whole numbers by multiples of 10 in the range 10–90 (e.g., 9 × 80, 5 × 60) using strategies based on place value and properties of operations.

Number and Operations—Fractions[5] 3.NF

Develop understanding of fractions as numbers.

1. Understand a fraction 1/*b* as the quantity formed by 1 part when a whole is partitioned into *b* equal parts; understand a fraction *a*/*b* as the quantity formed by *a* parts of size 1/*b*.
2. Understand a fraction as a number on the number line; represent fractions on a number line diagram.
    a. Represent a fraction 1/*b* on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into *b* equal parts. Recognize that each part has size 1/*b* and that the endpoint of the part based at 0 locates the number 1/*b* on the number line.
    b. Represent a fraction *a*/*b* on a number line diagram by marking off *a* lengths 1/*b* from 0. Recognize that the resulting interval has size *a*/*b* and that its endpoint locates the number *a*/*b* on the number line.
3. Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.
    a. Understand two fractions as equivalent (equal) if they are the same size, or the same point on a number line.
    b. Recognize and generate simple equivalent fractions, e.g., 1/2 = 2/4, 4/6 = 2/3). Explain why the fractions are equivalent, e.g., by using a visual fraction model.
    c. Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers. Examples: Express 3 in the form 3 = 3/1; recognize that 6/1 = 6; locate 4/4 and 1 at the same point of a number line diagram.
    d. Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

Measurement and Data 3.MD

Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.

1. Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes, e.g., by representing the problem on a number line diagram.
2. Measure and estimate liquid volumes and masses of objects using standard units of grams (g), kilograms (kg), and liters (l).[6] Add, subtract, multiply, or divide to solve one-step word problems

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[4] A range of algorithms may be used.
[5] Grade 3 expectations in this domain are limited to fractions with denominators 2, 3, 4, 6, and 8.
[6] Excludes compound units such as cm^3 and finding the geometric volume of a container.

involving masses or volumes that are given in the same units, e.g., by using drawings (such as a beaker with a measurement scale) to represent the problem.[7]

Represent and interpret data.

3. Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs. *For example, draw a bar graph in which each square in the bar graph might represent 5 pets.*
4. Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot, where the horizontal scale is marked off in appropriate units— whole numbers, halves, or quarters.

Geometric measurement: understand concepts of area and relate area to multiplication and to addition.

5. Recognize area as an attribute of plane figures and understand concepts of area measurement.
   a. A square with side length 1 unit, called "a unit square," is said to have "one square unit" of area, and can be used to measure area.
   b. A plane figure which can be covered without gaps or overlaps by *n* unit squares is said to have an area of *n* square units.
6. Measure areas by counting unit squares (square cm, square m, square in, square ft, and improvised units).
7. Relate area to the operations of multiplication and addition.
   a. Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths.
   b. Multiply side lengths to find areas of rectangles with whole-number side lengths in the context of solving real world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning.
   c. Use tiling to show in a concrete case that the area of a rectangle with whole-number side lengths a and $b + c$ is the sum of $a \times b$ and $a \times c$. Use area models to represent the distributive property in mathematical reasoning.
   d. Recognize area as additive. Find areas of rectilinear figures by decomposing them into non-overlapping rectangles and adding the areas of the non-overlapping parts, applying this technique to solve real world problems.

Geometric measurement: recognize perimeter as an attribute of plane figures and distinguish between linear and area measures.

8. Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.

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[7] Excludes multiplicative comparison problems (problems involving notions of "times as much"; see Glossary, Table 2).

3 Grade 3

Geometry 3.G

Reason with shapes and their attributes.

1. Understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attributes (e.g., having four sides), and that the shared attributes can define a larger category (e.g., quadrilaterals). Recognize rhombuses, rectangles, and squares as examples of quadrilaterals, and draw examples of quadrilaterals that do not belong to any of these subcategories.
2. Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole. *For example, partition a shape into 4 parts with equal area, and describe the area of each part as 1/4 of the area of the shape.*

Mathematics | Grade 4

In Grade 4, instructional time should focus on three critical areas: (1) developing understanding and fluency with multi-digit multiplication, and developing understanding of dividing to find quotients involving multi-digit dividends; (2) developing an understanding of fraction equivalence, addition and subtraction of fractions with like denominators, and multiplication of fractions by whole numbers; (3) understanding that geometric figures can be analyzed and classified based on their properties, such as having parallel sides, perpendicular sides, particular angle measures, and symmetry.

(1) Students generalize their understanding of place value to 1,000,000, understanding the relative sizes of numbers in each place. They apply their understanding of models for multiplication (equal-sized groups, arrays, area models), place value, and properties of operations, in particular the distributive property, as they develop, discuss, and use efficient, accurate, and generalizable methods to compute products of multi-digit whole numbers. Depending on the numbers and the context, they select and accurately apply appropriate methods to estimate or mentally calculate products. They develop fluency with efficient procedures for multiplying whole numbers; understand and explain why the procedures work based on place value and properties of operations; and use them to solve problems. Students apply their understanding of models for division, place value, properties of operations, and the relationship of division to multiplication as they develop, discuss, and use efficient, accurate, and generalizable procedures to find quotients involving multi-digit dividends. They select and accurately apply appropriate methods to estimate and mentally calculate quotients, and interpret remainders based upon the context.

(2) Students develop understanding of fraction equivalence and operations with fractions. They recognize that two different fractions can be equal (e.g., 15/9 = 5/3), and they develop methods for generating and recognizing equivalent fractions. Students extend previous understandings about how fractions are built from unit fractions, composing fractions from unit fractions, decomposing fractions into unit fractions, and using the meaning of fractions and the meaning of multiplication to multiply a fraction by a whole number.

(3) Students describe, analyze, compare, and classify two-dimensional shapes. Through building, drawing, and analyzing two-dimensional shapes, students deepen their understanding of properties of two-dimensional objects and the use of them to solve problems involving symmetry.

4 Grade 4 Overview

Operations and Algebraic Thinking

- Use the four operations with whole numbers to solve problems.
- Gain familiarity with factors and multiples.
- Generate and analyze patterns.

Number and Operations in Base Ten

- Generalize place value understanding for multi-digit whole numbers.
- Use place value understanding and properties of operations to perform multi-digit arithmetic.

Number and Operations—Fractions

- Extend understanding of fraction equivalence and ordering.
- Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.
- Understand decimal notation for fractions, and compare decimal fractions.

Measurement and Data

- Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.
- Represent and interpret data.
- Geometric measurement: understand concepts of angle and measure angles.

Geometry

- Draw and identify lines and angles, and classify shapes by properties of their lines and angles.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

4 Grade 4

Operations and Algebraic Thinking 4.OA

Use the four operations with whole numbers to solve problems.

1. Interpret a multiplication equation as a comparison, e.g., interpret 35 = 5 × 7 as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations.
2. Multiply or divide to solve word problems involving multiplicative comparison, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem, distinguishing multiplicative comparison from additive comparison.[1]
3. Solve multistep word problems posed with whole numbers and having whole-number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.

Gain familiarity with factors and multiples.

4. Find all factor pairs for a whole number in the range 1–100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1–100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1–100 is prime or composite.

Generate and analyze patterns.

5. Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself. *For example, given the rule "Add 3" and the starting number 1, generate terms in the resulting sequence and observe that the terms appear to alternate between odd and even numbers. Explain informally why the numbers will continue to alternate in this way.*

Number and Operations in Base Ten[2] 4.NBT

Generalize place value understanding for multi-digit whole numbers.

1. Recognize that in a multi-digit whole number, a digit in one place represents ten times what it represents in the place to its right. *For example, recognize that 700 ÷ 70 = 10 by applying concepts of place value and division.*
2. Read and write multi-digit whole numbers using base-ten numerals, number names, and expanded form. Compare two multi-digit numbers based on meanings of the digits in each place, using >, =, and < symbols to record the results of comparisons.
3. Use place value understanding to round multi-digit whole numbers to any place.

Use place value understanding and properties of operations to perform multi-digit arithmetic.

4. Fluently add and subtract multi-digit whole numbers using the standard algorithm.
5. Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

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[1] See Glossary, Table 2.

[2] Grade 4 expectations in this domain are limited to whole numbers less than or equal to 1,000,000.

4 Grade 4

6. Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

Number and Operations—Fractions[3] 4.NF

Extend understanding of fraction equivalence and ordering.

1. Explain why a fraction *a*/*b* is equivalent to a fraction (*n* × *a*)/(*n* × *b*) by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.
2. Compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as 1/2. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.

3. Understand a fraction *a*/*b* with *a* > 1 as a sum of fractions 1/*b*.
    a. Understand addition and subtraction of fractions as joining and separating parts referring to the same whole.
    b. Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation. Justify decompositions, e.g., by using a visual fraction model. *Examples: 3/8 = 1/8 + 1/8 + 1/8 ; 3/8 = 1/8 + 2/8 ; 2 1/8 = 1 + 1 + 1/8 = 8/8 + 8/8 + 1/8.*
    c. Add and subtract mixed numbers with like denominators, e.g., by replacing each mixed number with an equivalent fraction, and/or by using properties of operations and the relationship between addition and subtraction.
    d. Solve word problems involving addition and subtraction of fractions referring to the same whole and having like denominators, e.g., by using visual fraction models and equations to represent the problem.
4. Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.
    a. Understand a fraction *a*/*b* as a multiple of 1/*b*. *For example, use a visual fraction model to represent 5/4 as the product 5 × (1/4), recording the conclusion by the equation 5/4 = 5 × (1/4).*
    b. Understand a multiple of *a*/*b* as a multiple of 1/*b*, and use this understanding to multiply a fraction by a whole number. *For example, use a visual fraction model to express 3 × (2/5) as 6 × (1/5), recognizing this product as 6/5. (In general, n × (a/b) = (n × a)/b.)*
    c. Solve word problems involving multiplication of a fraction by a whole number, e.g., by using visual fraction models and equations to represent the problem. *For example, if each person at a party will eat 3/8 of a pound of roast beef, and there will be 5 people at the party, how many pounds of roast beef will be needed? Between what two whole numbers does your answer lie?*

[3] Grade 4 expectations in this domain are limited to fractions with denominators 2, 3, 4, 5, 6, 8, 10, 12, and 100.

4 Grade 4

Understand decimal notation for fractions, and compare decimal fractions.

5. Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100.[4] *For example, express 3/10 as 30/100, and add 3/10 + 4/100 = 34/100.*
6. Use decimal notation for fractions with denominators 10 or 100. *For example, rewrite 0.62 as 62/100; describe a length as 0.62 meters; locate 0.62 on a number line diagram.*
7. Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using **the number line or another** visual model. **CA**

Measurement and Data 4.MD

Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.

1. Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table. *For example, know that 1 ft is 12 times as long as 1 in. Express the length of a 4 ft snake as 48 in. Generate a conversion table for feet and inches listing the number pairs (1, 12), (2, 24), (3, 36), ...*
2. Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.
3. Apply the area and perimeter formulas for rectangles in real world and mathematical problems. *For example, find the width of a rectangular room given the area of the flooring and the length, by viewing the area formula as a multiplication equation with an unknown factor.*

Represent and interpret data.

4. Make a line plot to display a data set of measurements in fractions of a unit (1/2, 1/4, 1/8). Solve problems involving addition and subtraction of fractions by using information presented in line plots. *For example, from a line plot find and interpret the difference in length between the longest and shortest specimens in an insect collection.*

Geometric measurement: understand concepts of angle and measure angles.

5. Recognize angles as geometric shapes that are formed wherever two rays share a common endpoint, and understand concepts of angle measurement:
   a. An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through 1/360 of a circle is called a "one-degree angle," and can be used to measure angles.
   b. An angle that turns through *n* one-degree angles is said to have an angle measure of *n* degrees.

[4] Students who can generate equivalent fractions can develop strategies for adding fractions with unlike denominators in general. But addition and subtraction with unlike denominators in general is not a requirement at this grade.

4 Grade 4

6. Measure angles in whole-number degrees using a protractor. Sketch angles of specified measure.
7. Recognize angle measure as additive. When an angle is decomposed into non-overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Solve addition and subtraction problems to find unknown angles on a diagram in real world and mathematical problems, e.g., by using an equation with a symbol for the unknown angle measure.

Geometry 4.G

Draw and identify lines and angles, and classify shapes by properties of their lines and angles.

1. Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.
2. Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles. **(Two dimensional shapes should include special triangles, e.g., equilateral, isosceles, scalene, and special quadrilaterals, e.g., rhombus, square, rectangle, parallelogram, trapezoid.) CA**
3. Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry.

Mathematics | Grade 5

In Grade 5, instructional time should focus on three critical areas: (1) developing fluency with addition and subtraction of fractions, and developing understanding of the multiplication of fractions and of division of fractions in limited cases (unit fractions divided by whole numbers and whole numbers divided by unit fractions); (2) extending division to 2-digit divisors, integrating decimal fractions into the place value system and developing understanding of operations with decimals to hundredths, and developing fluency with whole number and decimal operations; and (3) developing understanding of volume.

(1) Students apply their understanding of fractions and fraction models to represent the addition and subtraction of fractions with unlike denominators as equivalent calculations with like denominators. They develop fluency in calculating sums and differences of fractions, and make reasonable estimates of them. Students also use the meaning of fractions, of multiplication and division, and the relationship between multiplication and division to understand and explain why the procedures for multiplying and dividing fractions make sense. (Note: this is limited to the case of dividing unit fractions by whole numbers and whole numbers by unit fractions.)

(2) Students develop understanding of why division procedures work based on the meaning of base-ten numerals and properties of operations. They finalize fluency with multi-digit addition, subtraction, multiplication, and division. They apply their understandings of models for decimals, decimal notation, and properties of operations to add and subtract decimals to hundredths. They develop fluency in these computations, and make reasonable estimates of their results. Students use the relationship between decimals and fractions, as well as the relationship between finite decimals and whole numbers (i.e., a finite decimal multiplied by an appropriate power of 10 is a whole number), to understand and explain why the procedures for multiplying and dividing finite decimals make sense. They compute products and quotients of decimals to hundredths efficiently and accurately.

(3) Students recognize volume as an attribute of three-dimensional space. They understand that volume can be measured by finding the total number of same-size units of volume required to fill the space without gaps or overlaps. They understand that a 1-unit by 1-unit by 1-unit cube is the standard unit for measuring volume. They select appropriate units, strategies, and tools for solving problems that involve estimating and measuring volume. They decompose three-dimensional shapes and find volumes of right rectangular prisms by viewing them as decomposed into layers of arrays of cubes. They measure necessary attributes of shapes in order to determine volumes to solve real-world and mathematical problems.

5 Grade 5 Overview

Operations and Algebraic Thinking

- Write and interpret numerical expressions.
- Analyze patterns and relationships.

Number and Operations in Base Ten

- Understand the place value system.
- Perform operations with multi-digit whole numbers and with decimals to hundredths.

Number and Operations—Fractions

- Use equivalent fractions as a strategy to add and subtract fractions.
- Apply and extend previous understandings of multiplication and division to multiply and divide fractions.

Measurement and Data

- Convert like measurement units within a given measurement system.
- Represent and interpret data.
- Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition.

Geometry

- Graph points on the coordinate plane to solve real-world and mathematical problems.
- Classify two-dimensional figures into categories based on their properties.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

5 Grade 5

Operations and Algebraic Thinking 5.OA

Write and interpret numerical expressions.

1. Use parentheses, brackets, or braces in numerical expressions, and evaluate expressions with these symbols.
2. Write simple expressions that record calculations with numbers, and interpret numerical expressions without evaluating them. *For example, express the calculation "add 8 and 7, then multiply by 2" as 2 × (8 + 7). Recognize that 3 × (18932 + 921) is three times as large as 18932 + 921, without having to calculate the indicated sum or product.*

2.1 Express a whole number in the range 2–50 as a product of its prime factors. For example, find the prime factors of 24 and express 24 as 2x2x2x3. CA

Analyze patterns and relationships.

3. Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane. *For example, given the rule "Add 3" and the starting number 0, and given the rule "Add 6" and the starting number 0, generate terms in the resulting sequences, and observe that the terms in one sequence are twice the corresponding terms in the other sequence. Explain informally why this is so.*

Number and Operations in Base Ten 5.NBT

Understand the place value system.

1. Recognize that in a multi-digit number, a digit in one place represents 10 times as much as it represents in the place to its right and 1/10 of what it represents in the place to its left.
2. Explain patterns in the number of zeros of the product when multiplying a number by powers of 10, and explain patterns in the placement of the decimal point when a decimal is multiplied or divided by a power of 10. Use whole-number exponents to denote powers of 10.
3. Read, write, and compare decimals to thousandths.
    a. Read and write decimals to thousandths using base-ten numerals, number names, and expanded form, e.g., 347.392 = 3 × 100 + 4 × 10 + 7 × 1 + 3 × (1/10) + 9 × (1/100) + 2 × (1/1000).
    b. Compare two decimals to thousandths based on meanings of the digits in each place, using >, =, and < symbols to record the results of comparisons.
4. Use place value understanding to round decimals to any place.

Perform operations with multi-digit whole numbers and with decimals to hundredths.

5. Fluently multiply multi-digit whole numbers using the standard algorithm.
6. Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.
7. Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

5 Grade 5

Number and Operations—Fractions 5.NF

Use equivalent fractions as a strategy to add and subtract fractions.

1. Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators. *For example, 2/3 + 5/4 = 8/12 + 15/12 = 23/12. (In general, a/b + c/d = (ad + bc)/bd.)*
2. Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators, e.g., by using visual fraction models or equations to represent the problem. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers. *For example, recognize an incorrect result 2/5 + 1/2 = 3/7, by observing that 3/7 < 1/2.*

Apply and extend previous understandings of multiplication and division to multiply and divide fractions.

3. Interpret a fraction as division of the numerator by the denominator (*a*/*b* = *a* ÷ *b*). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem. *For example, interpret 3/4 as the result of dividing 3 by 4, noting that 3/4 multiplied by 4 equals 3, and that when 3 wholes are shared equally among 4 people each person has a share of size 3/4. If 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get? Between what two whole numbers does your answer lie?*
4. Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.
    a. Interpret the product (*a*/*b*) × *q* as *a* parts of a partition of *q* into *b* equal parts; equivalently, as the result of a sequence of operations *a* × *q* ÷ *b*. *For example, use a visual fraction model to show (2/3) × 4 = 8/3, and create a story context for this equation. Do the same with (2/3) × (4/5) = 8/15. (In general, (a/b) × (c/d) = ac/bd.)*
    b. Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.
5. Interpret multiplication as scaling (resizing), by:
    a. Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication.
    b. Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence *a*/*b* = (*n* × *a*)/(*n* *b*) to the effect of multiplying *a*/*b* by 1.
6. Solve real world problems involving multiplication of fractions and mixed numbers, e.g., by using visual fraction models or equations to represent the problem.
7. Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.[1]

[1] Students able to multiply fractions in general can develop strategies to divide fractions in general, by reasoning about the relationship between multiplication and division. But division of a fraction by a fraction is not a requirement at this grade.

5 Grade 5

a. Interpret division of a unit fraction by a non-zero whole number, and compute such quotients. For example, create a story context for (1/3) ÷ 4, and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that (1/3) ÷ 4 = 1/12 because (1/12) × 4 = 1/3.
b. Interpret division of a whole number by a unit fraction, and compute such quotients. For example, create a story context for 4 ÷ (1/5), and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that 4 ÷ (1/5) = 20 because 20 × (1/5) = 4.
c. Solve real world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions, e.g., by using visual fraction models and equations to represent the problem. For example, how much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 1/3-cup servings are in 2 cups of raisins?

Measurement and Data 5.MD

Convert like measurement units within a given measurement system.

1. Convert among different-sized standard measurement units within a given measurement system (e.g., convert 5 cm to 0.05 m), and use these conversions in solving multi-step, real world problems.

Represent and interpret data.

2. Make a line plot to display a data set of measurements in fractions of a unit (1/2, 1/4, 1/8). Use operations on fractions for this grade to solve problems involving information presented in line plots. *For example, given different measurements of liquid in identical beakers, find the amount of liquid each beaker would contain if the total amount in all the beakers were redistributed equally.*

Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition.

3. Recognize volume as an attribute of solid figures and understand concepts of volume measurement.
    a. A cube with side length 1 unit, called a “unit cube,” is said to have “one cubic unit” of volume, and can be used to measure volume.
    b. A solid figure which can be packed without gaps or overlaps using n unit cubes is said to have a volume of n cubic units.
4. Measure volumes by counting unit cubes, using cubic cm, cubic in, cubic ft, and improvised units.
5. Relate volume to the operations of multiplication and addition and solve real world and mathematical problems involving volume.
    a. Find the volume of a right rectangular prism with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths, equivalently by multiplying the height by the area of the base. Represent threefold whole-number products as volumes, e.g., to represent the associative property of multiplication.
    b. Apply the formulas $V = l \times w \times h$ and $V = b \times h$ for rectangular prisms to find volumes of right rectangular prisms with whole-number edge lengths in the context of solving real world and mathematical problems.
    c. Recognize volume as additive. Find volumes of solid figures composed of two non-overlapping right rectangular prisms by adding the volumes of the non-overlapping parts, applying this technique to solve real world problems.

5 Grade 5

Geometry 5.G

Graph points on the coordinate plane to solve real-world and mathematical problems.

1. Use a pair of perpendicular number lines, called axes, to define a coordinate system, with the intersection of the lines (the origin) arranged to coincide with the 0 on each line and a given point in the plane located by using an ordered pair of numbers, called its coordinates. Understand that the first number indicates how far to travel from the origin in the direction of one axis, and the second number indicates how far to travel in the direction of the second axis, with the convention that the names of the two axes and the coordinates correspond (e.g., x-axis and x-coordinate, y-axis and y-coordinate).
2. Represent real world and mathematical problems by graphing points in the first quadrant of the coordinate plane, and interpret coordinate values of points in the context of the situation.

Classify two-dimensional figures into categories based on their properties.

3. Understand that attributes belonging to a category of two-dimensional figures also belong to all subcategories of that category. *For example, all rectangles have four right angles and squares are rectangles, so all squares have four right angles.*
4. Classify two-dimensional figures in a hierarchy based on properties.

Glossary

Addition and subtraction within 5, 10, 20, 100, or 1000. Addition or subtraction of two whole numbers with whole number answers, and with sum or minuend in the range 0-5, 0-10, 0-20, or 0-100, respectively. Example: 8 + 2 = 10 is an addition within 10, 14 – 5 = 9 is a subtraction within 20, and 55 – 18 = 37 is a subtraction within 100.

Additive inverses. Two numbers whose sum is 0 are additive inverses of one another. Example: 3/4 and – 3/4 are additive inverses of one another because 3/4 + (– 3/4) = (– 3/4) + 3/4 = 0.

Associative property of addition. See Table 3 in this Glossary.

Associative property of multiplication. See Table 3 in this Glossary.

Bivariate data. Pairs of linked numerical observations. Example: a list of heights and weights for each player on a football team.

Box plot. A method of visually displaying a distribution of data values by using the median, quartiles, and extremes of the data set. A box shows the middle 50% of the data.[1]

Commutative property. See Table 3 in this Glossary.

Complex fraction. A fraction *A*/*B* where *A* and/or *B* are fractions (*B* nonzero).

Computation algorithm. A set of predefined steps applicable to a class of problems that gives the correct result in every case when the steps are carried out correctly. *See also:* computation strategy.

Computation strategy. Purposeful manipulations that may be chosen for specific problems, may not have a fixed order, and may be aimed at converting one problem into another. *See also:* computation algorithm.

Congruent. Two plane or solid figures are congruent if one can be obtained from the other by rigid motion (a sequence of rotations, reflections, and translations).

Counting on. A strategy for finding the number of objects in a group without having to count every member of the group. For example, if a stack of books is known to have 8 books and 3 more books are added to the top, it is not necessary to count the stack all over again. One can find the total by

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[1] Adapted from Wisconsin Department of Public Instruction, http://dpi.wi.gov/standards/mathglos.html, accessed March 2, 2010.

counting on—pointing to the top book and saying "eight," following this with "nine, ten, eleven. There are eleven books now."

Dot plot. *See:* line plot.

Dilation. A transformation that moves each point along the ray through the point emanating from a fixed center, and multiplies distances from the center by a common scale factor.

Expanded form. A multi-digit number is expressed in expanded form when it is written as a sum of single-digit multiples of powers of ten. For example, 643 = 600 + 40 + 3.

Expected value. For a random variable, the weighted average of its possible values, with weights given by their respective probabilities.

First quartile. For a data set with median *M*, the first quartile is the median of the data values less than *M*. Example: For the data set {1, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the first quartile is 6.[2] *See also:* median, third quartile, interquartile range.

Fraction. A number expressible in the form *a*/*b* where *a* is a whole number and *b* is a positive whole number. (The word *fraction* in these standards always refers to a non-negative number.) *See also:* rational number.

Identity property of 0. See Table 3 in this Glossary.

Independently combined probability models. Two probability models are said to be combined independently if the probability of each ordered pair in the combined model equals the product of the original probabilities of the two individual outcomes in the ordered pair.

Integer. A number expressible in the form *a* or –*a* for some whole number *a*.

Interquartile Range. A measure of variation in a set of numerical data, the interquartile range is the distance between the first and third quartiles of the data set. Example: For the data set {1, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the interquartile range is 15 – 6 = 9. *See also:* first quartile, third quartile.

Line plot. A method of visually displaying a distribution of data values where each data value is shown as a dot or mark above a number line. Also known as a dot plot.[3]

[2] Many different methods for computing quartiles are in use. The method defined here is sometimes called the Moore and McCabe method. See Langford, E., "Quartiles in Elementary Statistics," *Journal of Statistics Education* Volume 14, Number 3 (2006).
[3] Adapted from Wisconsin Department of Public Instruction, *op. cit*.

Mean. A measure of center in a set of numerical data, computed by adding the values in a list and then dividing by the number of values in the list.[4] Example: For the data set {1, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the mean is 21.

Mean absolute deviation. A measure of variation in a set of numerical data, computed by adding the distances between each data value and the mean, then dividing by the number of data values. Example: For the data set {2, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the mean absolute deviation is 20.

Median. A measure of center in a set of numerical data. The median of a list of values is the value appearing at the center of a sorted version of the list—or the mean of the two central values, if the list contains an even number of values. Example: For the data set {2, 3, 6, 7, 10, 12, 14, 15, 22, 90}, the median is 11.

Midline. In the graph of a trigonometric function, the horizontal line halfway between its maximum and minimum values.

Multiplication and division within 100. Multiplication or division of two whole numbers with whole number answers, and with product or dividend in the range 0-100. Example: 72 ÷ 8 = 9.

Multiplicative inverses. Two numbers whose product is 1 are multiplicative inverses of one another. Example: 3/4 and 4/3 are multiplicative inverses of one another because 3/4 × 4/3 = 4/3 × 3/4 = 1.

Number line diagram. A diagram of the number line used to represent numbers and support reasoning about them. In a number line diagram for measurement quantities, the interval from 0 to 1 on the diagram represents the unit of measure for the quantity.

Percent rate of change. A rate of change expressed as a percent. Example: if a population grows from 50 to 55 in a year, it grows by 5/50 = 10% per year.

Probability distribution. The set of possible values of a random variable with a probability assigned to each.

Properties of operations. See Table 3 in this Glossary.

Properties of equality. See Table 4 in this Glossary.

Properties of inequality. See Table 5 in this Glossary.

Properties of operations. See Table 3 in this Glossary.

[4] To be more precise, this defines the *arithmetic mean*.

Probability. A number between 0 and 1 used to quantify likelihood for processes that have uncertain outcomes (such as tossing a coin, selecting a person at random from a group of people, tossing a ball at a target, or testing for a medical condition).

Probability model. A probability model is used to assign probabilities to outcomes of a chance process by examining the nature of the process. The set of all outcomes is called the sample space, and their probabilities sum to 1. *See also:* uniform probability model.

Random variable. An assignment of a numerical value to each outcome in a sample space.

Rational expression. A quotient of two polynomials with a non-zero denominator.

Rational number. A number expressible in the form a/b or $-a/b$ for some fraction a/b. The rational numbers include the integers.

Rectilinear figure. A polygon all angles of which are right angles.

Rigid motion. A transformation of points in space consisting of a sequence of one or more translations, reflections, and/or rotations. Rigid motions are here assumed to preserve distances and angle measures.

Repeating decimal. The decimal form of a rational number. *See also:* terminating decimal.

Sample space. In a probability model for a random process, a list of the individual outcomes that are to be considered.

Scatter plot. A graph in the coordinate plane representing a set of bivariate data. For example, the heights and weights of a group of people could be displayed on a scatter plot.[5]

Similarity transformation. A rigid motion followed by a dilation.

Tape diagram. A drawing that looks like a segment of tape, used to illustrate number relationships. Also known as a strip diagram, bar model, fraction strip, or length model.

Terminating decimal. A decimal is called terminating if its repeating digit is 0.

Third quartile. For a data set with median M, the third quartile is the median of the data values greater than M. Example: For the data set {2, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the third quartile is 15. *See also:* median, first quartile, interquartile range.

[5] Adapted from Wisconsin Department of Public Instruction, *op. cit*.

Transitivity principle for indirect measurement. If the length of object A is greater than the length of object B, and the length of object B is greater than the length of object C, then the length of object A is greater than the length of object C. This principle applies to measurement of other quantities as well.

Uniform probability model. A probability model which assigns equal probability to all outcomes. *See also:* probability model*.*

Vector. A quantity with magnitude and direction in the plane or in space, defined by an ordered pair or triple of real numbers.

Visual fraction model. A tape diagram, number line diagram, or area model.

Whole numbers. The numbers 0, 1, 2, 3, ….

Table 1. Common addition and subtraction situations.[6]

	Result Unknown	Change Unknown	Start Unknown
Add to	Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now? 2 + 3 = ?	Two bunnies were sitting on the grass. Some more bunnies hopped there. Then there were five bunnies. How many bunnies hopped over to the first two? 2 + ? = 5	Some bunnies were sitting on the grass. Three more bunnies hopped there. Then there were five bunnies. How many bunnies were on the grass before? ? + 3 = 5
Take from	Five apples were on the table. I ate two apples. How many apples are on the table now? 5 – 2 = ?	Five apples were on the table. I ate some apples. Then there were three apples. How many apples did I eat? 5 – ? = 3	Some apples were on the table. I ate two apples. Then there were three apples. How many apples were on the table before? ? – 2 = 3

	Total Unknown	Addend Unknown	Both Addends Unknown[7]
Put Together/ Take Apart[8]	Three red apples and two green apples are on the table. How many apples are on the table? 3 + 2 = ?	Five apples are on the table. Three are red and the rest are green. How many apples are green? 3 + ? = 5, 5 – 3 = ?	Grandma has five flowers. How many can she put in her red vase and how many in her blue vase? 5 = 0 + 5, 5 = 5 + 0 5 = 1 + 4, 5 = 4 + 1 5 = 2 + 3, 5 = 3 + 2

	Difference Unknown	Bigger Unknown	Smaller Unknown
Compare[9]	("How many more?" version): Lucy has two apples. Julie has five apples. How many more apples does Julie have than Lucy? ("How many fewer?" version): Lucy has two apples. Julie has five apples. How many fewer apples does Lucy have than Julie? 2 + ? = 5, 5 – 2 = ?	(Version with "more"): Julie has three more apples than Lucy. Lucy has two apples. How many apples does Julie have? (Version with "fewer"): Lucy has 3 fewer apples than Julie. Lucy has two apples. How many apples does Julie have? 2 + 3 = ?, 3 + 2 = ?	(Version with "more"): Julie has three more apples than Lucy. Julie has five apples. How many apples does Lucy have? (Version with "fewer"): Lucy has 3 fewer apples than Julie. Julie has five apples. How many apples does Lucy have? 5 – 3 = ?, ? + 3 = 5

[6] Adapted from Boxes 2–4 of *Mathematics Learning in Early Childhood*, National Research Council (2009, pp. 32–33).

[7] These take apart situations can be used to show all the decompositions of a given number. The associated equations, which have the total on the left of the equal sign, help children understand that the = sign does not always mean *makes* or *results in* but always does mean *is the same number as*.

[8] Either addend can be unknown, so there are three variations of these problem situations. Both Addends Unknown is a productive extension of this basic situation, especially for small numbers less than or equal to 10.

[9] For the Bigger Unknown or Smaller Unknown situations, one version directs the correct operation (the version using *more* for the bigger unknown and using *less* for the smaller unknown). The other versions are more difficult.

Table 2. Common multiplication and division situations.[10]

	Unknown Product $3 \times 6 = ?$	Group Size Unknown ("How many in each group?" Division) $3 \times ? = 18$ and $18 \div 3 = ?$	Number of Groups Unknown ("How many groups?" Division) $? \times 6 = 18$ and $18 \div 6 = ?$
Equal Groups	There are 3 bags with 6 plums in each bag. How many plums are there in all? *Measurement example*. You need 3 lengths of string, each 6 inches long. How much string will you need altogether?	If 18 plums are shared equally into 3 bags, then how many plums will be in each bag? *Measurement example*. You have 18 inches of string, which you will cut into 3 equal pieces. How long will each piece of string be?	If 18 plums are to be packed 6 to a bag, then how many bags are needed? *Measurement example*. You have 18 inches of string, which you will cut into pieces that are 6 inches long. How many pieces of string will you have?
Arrays,[11] Area[12]	There are 3 rows of apples with 6 apples in each row. How many apples are there? *Area example*. What is the area of a 3 cm by 6 cm rectangle?	If 18 apples are arranged into 3 equal rows, how many apples will be in each row? *Area example*. A rectangle has area 18 square centimeters. If one side is 3 cm long, how long is a side next to it?	If 18 apples are arranged into equal rows of 6 apples, how many rows will there be? *Area example*. A rectangle has area 18 square centimeters. If one side is 6 cm long, how long is a side next to it?
Compare	A blue hat costs $6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost? *Measurement example*. A rubber band is 6 cm long. How long will the rubber band be when it is stretched to be 3 times as long?	A red hat costs $18 and that is 3 times as much as a blue hat costs. How much does a blue hat cost? *Measurement example*. A rubber band is stretched to be 18 cm long and that is 3 times as long as it was at first. How long was the rubber band at first?	A red hat costs $18 and a blue hat costs $6. How many times as much does the red hat cost as the blue hat? *Measurement example*. A rubber band was 6 cm long at first. Now it is stretched to be 18 cm long. How many times as long is the rubber band now as it was at first?
General	$a \times b = ?$	$a \times ? = p$ and $p \div a = ?$	$? \times b = p$ and $p \div b = ?$

[10] The first examples in each cell are examples of discrete things. These are easier for students and should be given before the measurement examples.

[11] The language in the array examples shows the easiest form of array problems. A harder form is to use the terms rows and columns: The apples in the grocery window are in 3 rows and 6 columns. How many apples are in there? Both forms are valuable.

[12] Area involves arrays of squares that have been pushed together so that there are no gaps or overlaps, so array problems include these especially important measurement situations.

Table 3. The properties of operations.

Here *a*, *b* and *c* stand for arbitrary numbers in a given number system. The properties of operations apply to the rational number system, the real number system, and the complex number system.

Associative property of addition	$(a + b) + c = a + (b + c)$
Commutative property of addition	$a + b = b + a$
Additive identity property of 0	$a + 0 = 0 + a = a$
Existence of additive inverses	For every a there exists $-a$ so that $a + (-a) = (-a) + a = 0$.
Associative property of multiplication	$(a \times b) \times c = a \times (b \times c)$
Commutative property of multiplication	$a \times b = b \times a$
Multiplicative identity property of 1	$a \times 1 = 1 \times a = a$
Existence of multiplicative inverses	For every $a \neq 0$ there exists $1/a$ so that $a \times 1/a = 1/a \times a = 1$.
Distributive property of multiplication over addition	$a \times (b + c) = a \times b + a \times c$

Table 4. The properties of equality.

Here *a*, *b*, and *c* stand for arbitrary numbers in the rational, real, or complex number systems.

Reflexive property of equality	$a = a$
Symmetric property of equality	If $a = b$, then $b = a$.
Transitive property of equality	If $a = b$ and $b = c$, then $a = c$.
Addition property of equality	If $a = b$, then $a + c = b + c$.
Subtraction property of equality	If $a = b$, then $a - c = b - c$.
Multiplication property of equality	If $a = b$, then $a \times c = b \times c$.
Division property of equality	If $a = b$ and $c \neq 0$, then $a \div c = b \div c$.
Substitution property of equality	If $a = b$, then b may be substituted for a in any expression containing a.

Table 5. The properties of inequality.

Here a, b, and c stand for arbitrary numbers in the rational or real number systems.

Exactly one of the following is true: $a < b$, $a = b$, $a > b$.
If $a > b$ and $b > c$ then $a > c$.
If $a > b$, then $b < a$.
If $a > b$, then $-a < -b$.
If $a > b$, then $a \pm c > b \pm c$.
If $a > b$ and $c > 0$, then $a \times c > b \times c$.
If $a > b$ and $c < 0$, then $a \times c < b \times c$.
If $a > b$ and $c > 0$, then $a \div c > b \div c$.
If $a > b$ and $c < 0$, then $a \div c < b \div c$.

The Alignment of
the California Preschool Learning Foundations with Key Early Education Resources

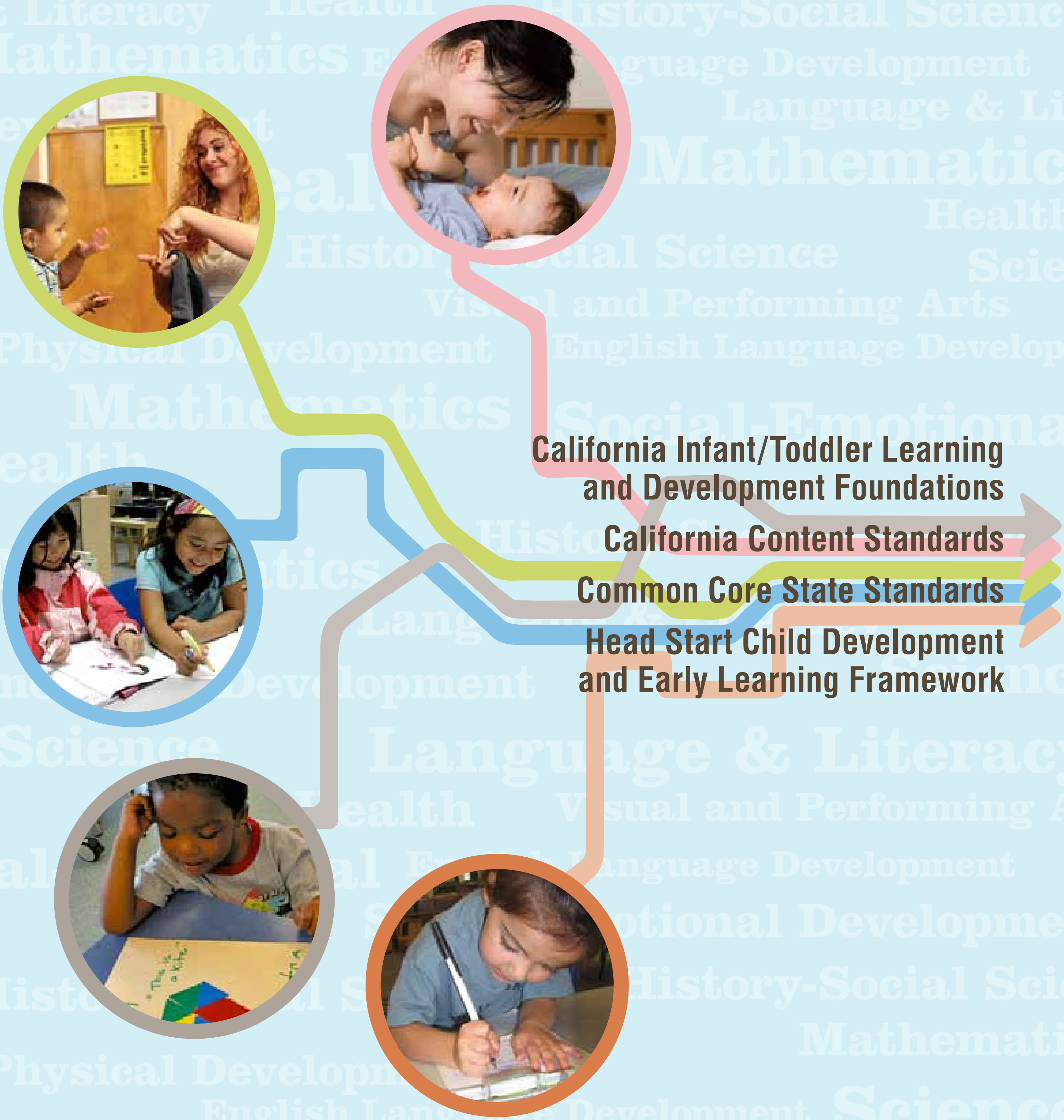

Table 1.8
Overview of the Alignment Between the Mathematics Domain and the Common Core State Standards

California Infant/Toddler Learning and Development Foundations	California Preschool Learning Foundations	Common Core State Standards Kindergarten
Cognitive Development	**Mathematics**	**Mathematics**
	Number Sense	**Counting and Cardinality**
Number Sense →	Children understand numbers and quantities in their everyday environment. →	Know number names and the count sequence Count to tell the number of objects Compare numbers
	Children understand number relationships and operations in their everyday environment. →	**Operations and Algebraic Thinking** Understand addition as putting together and adding to, and subtraction as taking apart and taking from **Number and Operations in Base Ten** Work with numbers 11–19 to gain foundations for place value
	Algebra and Functions (Classification and Patterning)	**Measurement and Data**
Classification →	Children sort and classify objects in their everyday environment. →	Classify objects and count the number of objects in categories
Understanding of Personal Routine →	Children recognize/expand understanding of simple repeating patterns.	

Table 1.8 *(continued)*

Mathematics		
California Infant/Toddler Learning and Development Foundations	**California Preschool Learning Foundations**	**Common Core State Standards Kindergarten**
Cognitive Development	**Mathematics**	**Mathematics**
	Measurement	**Measurement and Data**
Spatial Relationships →	Children compare, order, and measure objects. →	Describe and compare measurable attributes
	Geometry	**Geometry**
Spatial Relationships →	Children identify and use shapes. →	Identify and describe shapes (squares, circles, triangles, rectangles, hexagons, cubes, cones, cylinders, and spheres). Analyze, compare, create, and compose shapes.
	Children understand positions in space. →	Identify and describe shapes (squares, circles, triangles, rectangles, hexagons, cubes, cones, cylinders, and spheres).

Table 1.8 *(continued)*

California Infant/Toddler Learning and Development Foundations	California Preschool Learning Foundations	Common Core State Standards Kindergarten
Cognitive Development	**Mathematics**	**Mathematics**
Problem Solving →	Children use mathematical thinking to solve problems in their everyday environment. →	Make sense of problems and persevere in solving them. Reason abstractly and quantitatively. Construct viable arguments and critique the reasoning of others. Model with mathematics. Use appropriate tools strategically. Attend to precision. Look for and make use of structure. Look for and express regularity in repeated reasoning.

Table 1.9 Detailed View of the Alignment Between the Mathematics Domain and the Common Core State Standards		
California Preschool Learning Foundations		**Common Core State Standards** **Kindergarten**
Domain: **Mathematics**		**Mathematics**
Strand: **Number Sense**		**Counting and Cardinality** **Operations and Algebraic Thinking** **Number and Operations in Base Ten**
At around 48 months	**At around 60 months**	**By the end of kindergarten**
1.0 Children begin to understand numbers and quantities in their everyday environment.	**1.0 Children expand their understanding of numbers and quantities in their everyday environment.**	**Counting and Cardinality** ▪ **Know number names and the count sequence.** ▪ **Count to tell the number of objects.**
1.1 Recite numbers in order to ten with increasing accuracy.	1.1 Recites numbers in order to twenty with increasing accuracy.	**Know number names and the count sequence.** 1. Count to 100 by ones and by tens. 2. Count forward beginning from a given number within the known sequence (instead of having to begin at 1).
1.2 Begin to recognize and name a few written numerals.	1.2 Recognize and know the name of some written numerals.	**Know number names and the count sequence.** 3. Write numbers from 0 to 20. Represent a number of objects with a written numeral 0-20 (with 0 representing a count of no objects).
1.3 Identify, without counting, the number of objects in a collection of up to three objects (i.e., subitize).	1.3 Identify without counting the number of objects in a collection of up to four objects (i.e., subitize).	

1.4 Count up to five objects, using one-to-one correspondence (one object for each number word) with increasing accuracy.	1.4 Count up to ten objects, using one-to-one correspondence with increasing accuracy.	**Count to tell the number of objects.** 4. Understand the relationship between numbers and quantities; connect counting to cardinality. a. When counting objects, say the number names in the standard order, pairing each object with one and only one number name and each name with one and only one object. 5. Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects.
1.5 Use the number name of the last object counted to answer the question, "How many…?"	1.5 Understand, when counting, that the number name of the last object counted represent the total number of objects in the group (i.e., cardinality)	**Count to tell the number of objects.** 4. Understand the relationship between numbers and quantities; connect counting to cardinality. b. Understand that the last number name said tells the number of objects counted. The number of objects is the same regardless of their arrangement or the order in which they were counted. c. Understand that each successive number name refers to a quantity that is one larger.
2.0 Children begin to understand number relationships and operations in their everyday environment.	**2.0 Children expand their understanding of number relationships and operations in their everyday environment.**	**Counting and Cardinality** ▪ **Compare Numbers** **Operations and Algebraic Thinking** ▪ **Understand addition as putting and adding to, and understand subtraction as taking apart and taking from.**
At around 48 months	**At around 60 months**	**By the end of kindergarten**
2.1 Compare visually (with or without counting) two groups of objects that are obviously equal or nonequal and communicate, "more" or "same"	2.1 Compare, by counting or matching, two groups of up to five objects and communicate, "more," "same as," or "fewer" (or "less").	**Compare Numbers** 6. Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies. 7. Compares two numbers between 1 and 10 presented as written

		numerals.
2.2 Understand that adding to (or taking away) one or more objects from a group will increase (or decrease) the number of objects in the group.	2.2 Understands that adding one or taking away one changes the number in a small group of objects by exactly one.	**Understand addition as putting and adding to, and understand subtraction as taking apart and taking from.** 1. Represent addition and subtraction with objects, fingers, mental images, drawings, sounds (e.g., clap), acting out situations, verbal explanations, expressions, or equations. 2. Solve addition and subtraction word problems, and add and subtract within 10, e.g., by using objects or drawings to represent the problems. 3. Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., 5 = 2 + 3 and 5 = 4 + 1). 4. For any number from 1 to 9, find the number that makes 10 when added to the given number, e.g., by using objects or drawings, and record the answer with a drawing or equation. 5. Fluently add and subtract within 5.
2.3 Understand that putting two groups of objects together will make a bigger group.	2.3 Understand that putting two groups of objects together will make a bigger group and that a group of objects can be taken apart into smaller groups.	
2.4 Solve simple addition and subtraction problems nonverbally (and often verbally) with a very small number of objects (sums up to 4 or 5).	2.4 Solve simple addition and subtraction problems with a small number of objects (sums up to 10), usually by counting.	
		Number and Operations in Base Ten ▪ **Work with numbers 11–19 to gain foundations for place value.**
		Work with numbers 11–19 to gain foundations for place value. 1. Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (such as 18 = 10 + 8); understand that these numbers are composed by ten ones and one, two, three, four, five, six, seven, eight, or nine ones.

Strand: **Algebra and Functions (Classification and Patterning)**		**Measurement and Data**
At around 48 months	**At around 60 months**	**By the end of kindergarten**
1.0 Children begin to sort and classify objects in their everyday environment.	**1.0 Children expand their understanding of sorting and classifying objects in their everyday environment.**	**Measurement and Data** ▪ **Classify objects and count the number of objects in each category.**
1.1 Sort and classify objects by one attribute into two or more groups, with increasing accuracy.	1.1 Sort and classify objects by one or more attributes, into two or more groups, with increasing accuracy (e.g., may sort first by one attribute and then by another attribute).*	3. Classify objects into given categories, count the numbers of objects in each category and sort the categories by count.
2.0 Children begin to recognize simple, repeating patterns.	**2.0 Children expand their understanding of simple, repeating patterns.**	
2.1 Begin to identify or recognize a simple repeating pattern.	2.1 Recognizes and duplicates simple repeating patterns.	
2.2 Attempt to create a simple repeating pattern or participate in making one.	2.2 Begin to extend and create simple repeating patterns	

*The footnote that appears in the published version of this foundation has been omitted so that the alignment can be highlighted.

Strand: **Measurement**		**Measurement and Data**
At around 48 months	**At around 60 months**	**By the end of kindergarten**
1.0 Children begin to compare and order objects.	**1.0 Children expand their understanding of comparing, ordering, and measuring objects.**	**Measurement and Data** ▪ **Describe and compare measurable attributes.**
1.1 Demonstrate awareness that objects can be compared by length, weight, or capacity, by noting gross differences, using words such as bigger, longer, heavier, or taller, or by placing objects side by side to compare length.	1.1 Compare two objects by length, weight, or capacity directly (e.g., putting objects side by side) or indirectly (e.g., using a third object).	1. Describe measurable attributes of objects, such as length or weight. Describe several measurable attributes of a single object. 2. Directly compare two objects with a measurable attribute in common, to see which object has "more of"/ "less of" the attribute, and describe the difference. *For example, directly compare the heights of two children and describe one child as taller/shorter.*
1.2 Order three objects by size.	1.2 Order four or more objects by size.	
	1.3 Measure length using multiple duplicates of the same-size concrete units laid end to end.*	
		4. Demonstrate an understanding of concepts time (e.g., morning, afternoon, evening, today, yesterday, tomorrow, week, year) and tools that measure time (e.g., clock, calendar). (CA-Standard MG 1.2) ***a. Name the days of the week. (CA-Standard 1.3)*** ***b. Identify the time (to the nearest hour) of everyday events (e.g., lunch time is 12 o'clock, bedtime is 8 o'clock at night). (CA-Standard MG 1.4).***

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Strand: Geometry		Geometry
At around 48 months	**At around 60 months**	**By the end of kindergarten**
1.0 Children begin to identify and use common shapes in their everyday environment.	**1.0 Children identify and use a variety of shapes in their everyday environment.**	**Geometry** ▪ **Identify and describe shapes (squares, circles, triangles, hexagons, cubes, cones, cylinders, and spheres).** ▪ **Analyze, compare, create, and compose shapes.**
1.1 Identify simple two-dimensional shapes, such as a circle and square.	1.1 Identify, describe, and construct a variety of different shapes, including variations of circle, triangle, rectangle, square, and other shapes.	**Identify and describe shapes (squares, circles, triangles, hexagons, cubes, cones, cylinders, and spheres).** 1. Describe objects I the environment using names of shapes, and describe the relative positions of these objects using terms such as *above, below, beside, in front of, behind,* and *next to*. 2. Correctly name shapes regardless of their orientations or overall size. 3. Identify shapes as two-dimensional (lying in a plane, “flat”) or three-dimensional (“solid”). **Analyze, compare, create, and compose shapes.** 4. Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts (e.g., number of sides and vertices/”corners”) and other attributes (e.g., having sides of equal length).
1.2 Use individual shapes to represent different elements of a picture or design.	1.2 Combine different shapes to create a picture design.	**Analyze, compare, create, and compose shapes.** 5. Model shapes in the world by building shapes from components (e.g., sticks and clay balls) and drawing shapes. 6. Compose simple shapes to form larger shapes. *For example, “Can you join these triangles with full sides touching to make a rectangle?”*

2.0 Children begin to understand positions in space.	2.0 Children expand their understanding of positions in space.	Geometry ▪ Identify and describe shapes (squares, circles, triangles, hexagons, cubes, cones, cylinders, and spheres).
2.1 Identify positions of objects and people in space, such as in/on/under, up/down, and inside/outside.	2.1 Identify positions of objects and people in space, including in/on/under, up/down, inside/outside, beside/between, and in front/behind.	**Identify and describe shapes (squares, circles, triangles, hexagons, cubes, cones, cylinders, and spheres).** 1. Describe objects in the environment using names of shapes, and describe the relative positions of these objects using terms such as *above, below, beside, in front of, behind,* and *next to*.
Strand: **Mathematical Reasoning**		**Mathematical Practices**
At around 48 months	**At around 60 months**	**By the end of kindergarten**
1.0 Children use mathematical thinking to solve problems that arise in their everyday environment.	**1.0 Children expand the use of mathematical thinking to solve problems that arise in their everyday environment.**	**Mathematical Practices**
1.1 Begin to apply simple mathematical strategies to solve problems in their environment.	1.1 Identify and apply a variety of mathematical strategies to solve problems in their environment.	**Mathematical Practices** 1. Make sense of problems and persevere in solving them. 2. Reason abstractly and quantitatively. 3. Construct viable arguments and critique the reasoning of others. 4. Model with mathematics. 5. Use appropriate tools strategically. 6. Attend to precision. 7. Look for and make use of structure. 8. Look for and express regularity in repeated reasoning.